Some General Results on Perfect Domination Polynomials

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Abstract: *This paper introduces the concept of perfect domination polynomial of a graph. Perfect domination polynomial depends on the cardinality of perfect dominating sets of the graph. Results on the Perfect domination polynomial of certain classes of graphs are also obtained.*

Keywords: Perfect Dominating set, Double star Graph, Perfect Domination Polynomial

1. Introduction

Let $G = (V, E)$ be a simple graph of order n. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V \setminus uv \in E\}$. A set $S \subset V$ is a dominating set of G, if every vertex $u \in V$ is an element of S or is adjacent to an element of S [4]. The dominating set S is a perfect dominating set if $| N(u) \cap S |= 1$ for each $u \in V - S$, or equivalently, if every vertex $u \in V - S$ is adjacent to exactly one vertex in S [7]. Perfect dominating set is a subset of dominating set. This concept of Perfect domination was first studied by Weichsel. The perfect domination number γ_{pf} is the minimum cardinality of a perfect dominating set in G. Let $D_{pf}(G,i)$ be the family of perfect dominating sets of a graph G with cardinality i and let $d_{pf}(G,i) = \mid D_{pf}(G,i) \mid$. We call the polynomial $D_{pf}(G,x) = \sum_{i=\gamma_{pf}(G)}^{|V(G)|} d_{pf}(G,i)x^i$, the perfect domination polynomial of a graph G.

2. Perfect domination polynomial of a graph

Definition 2.1. Let $D_{pf}(G,i)$ be the family of perfect dominating sets of a graph G with Cardinality i and let $d_{pf}(G,i) = \mid D_{pf}(G,i) \mid$. Then the perfect Domination Polynomial $D_{pf}(G,x)$ of G is defined as $D_{pf}(G,x) = \sum_{i=\gamma_{pf}(G)}^{|V(G)|} d_{pf}(G,i)x^i$

Example 2.2. Consider the following graph G with five vertices in Figure 2.1

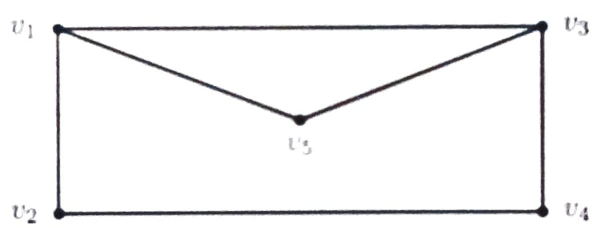

Figure 2.1. Graph in Example 2.2

The Perfect Dominating Sets of cardinality one is empty, which implies $\mid D_{pf}(G,1) \mid = 0$

The Perfect Dominating Sets of Cardinality two are $\{v_1, v_2\}$ and $\{v_3, v_4\}$, then

$\mid D_{pf}(G,2) \mid = 2$

The Perfect Dominating Sets of Cardinality three is $\{v_1, v_3, v_5\}$, then $\mid D_{pf}(G,3) \mid = 1$

The Perfect Dominating Sets of Cardinality four is empty, then $\mid D_{pf}(G,4) \mid = 0$

The Perfect Dominating Sets of Cardinality five is $\{v_1, v_2, v_3, v_4, v_5\}$ which implies $| D_{pf}(G, 5) |= 1$

Thus, $d_{pf}(G, 1) = 0, d_{pf}(G, 2) = 2, d_{pf}(G, 3) = 1, d_{pf}(G, 4) = 0 and d_{pf}(G, 5) = 1$.

Note that $\gamma_{pf}(G) = 2$. Hence,

$$\begin{aligned} D_{pf}(G, x) &= \sum_{i=2}^{5} d_{pf}(G, i)x^i \\ &= d_{pf}(G, 2)x^2 + d_{pf}(G, 3)x^3 + d_{pf}(G, 5)x^5 \\ &= 2x^2 + x^3 + x^5 \end{aligned}$$

Theorem 2.3. Let G be a graph of two components G_1 and G_2 then $D_{pf}(G, x) = D_{pf}(G_1, x)D_pf(G_2, x)$

Proof. Given G be a graph consists of two components G_1 and G_2. Let $\gamma_{pf}(G_1)$ and $\gamma_{pf}(G_2)$ be the perfect dominating number for the graphs G_1 and G_2 respectively. Let $j \in \{\gamma_{pf}(G_1), \gamma_{pf}(G_1 + 1), \ldots, | V(G_1) |\}$. A Perfect Dominating Set of k vertices in G arises by choosing a Perfect Dominating set of j vertices in G_1 and a perfect dominating set of $k - j$ vertices in G_2. Doing this same process for all j we get exactly the coefficient of x^k in $D_{pf}(G_1, x)D_pf(G_2, x)$. Hence, we get the same coefficient of x^k in $D_{pf}(G, x)$ and $D_{pf}(G_1, x)D_{pf}(G_2, x)$. Therefore, $D_{pf}(G, x) = D_{pf}(G_1, x)D_{pf}(G_2, x)$. ∎

Example 2.4. Consider the Following Graph G with two components G_1 and G_2 as in Figure 2.2.

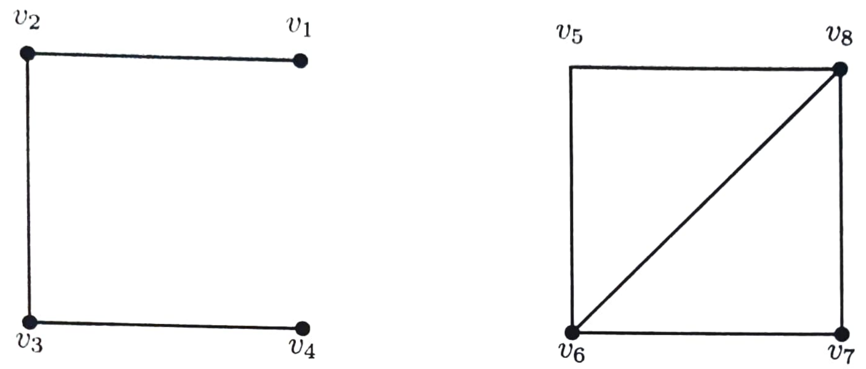

Figure 2.2. G_1 and G_2 of Example 2.4

First the perfect domination polynomials of G_1 and G_2 are to be found. Since, G_1 is a Path with four vertices $\gamma_{pf}(G_1) = \ulcorner 4/3 \urcorner = 2$ by Using Lemma 4.1.3. Now,

$$D_{pf}(G_1, 2) = \{\{v_1, v_4\}, \{v_2, v_3\}\} \implies, d_{pf}(G_1, 2) = \mid D_{pf}(G_1, 2) \mid = 2$$

$$D_{pf}(G_1, 3) = \{\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}\} \implies d_{pf}(G_1, 3) = \mid D_{pf}(G_1, 3) \mid = 2$$

$$D_{pf}(G_1, 4) = \{\{v_1, v_2, v_3, v_4\}\} \implies, d_{pf}(G_1, 4) = \mid D_{pf}(G_1, 4) \mid = 1$$

Therefore, the perfect domination polynomial of G_1 is given by

$$D_{pf}(G_1, x) = \sum_{i=2}^{4} d_{pf}(G_1, i)x^i = 2x^2 + 2x^3 + x^4$$

Now we find the Perfect domination polynomial of G_2. Clearly we have $\gamma_{pf}(G_2) = 1$ and $D_{pf}(G_2, 1) = \{\{v_6\}, \{v_8\}\}$. Therefore, $d_{pf}(G_2, 1) = \mid D_{pf}(G_2, 1) \mid = 2$. Also, $D_{pf}(G_2, 2)$ $= \phi$ and $D_{pf}(G_2, 3) = \phi$ hence, $d_{pf}(G_2, 2) = d_{pf}(G_2, 3) = 0$

Now, $D_{pf}(G_2, 4) = \{\{v_5, v_6, v_7, v_8\}\}$ hence, $d_{pf}(G_2, 4) = \mid D_{pf}(G_2, 4) \mid = 1$.

Therefore, the Perfect Domination Polynomial of G_2 is given by

$$D_{pf}(G_2, x) = \sum_{i=1}^{4} d_{pf}(G_2, i)x^i = 2x + x^4.$$

$$\begin{aligned} D_{pf}(G_1, x)D_{pf}(G_2, x) &= (2x^2 + 2x^3 + x^4)(2x + x^4) \\ &= 4x^3 + 4x^4 + 2x^5 + 2x^6 + 2x^7 + x^8 \qquad (1) \end{aligned}$$

Now, the perfect domination polynomial of G is to be found. Since, $\gamma_{pf}(G_1) = 2$ and $\gamma_{pf}(G_2) = 1$ therefore $\gamma_{pf}(G) = 3$, because G_1 and G_2 are the components of G.

$D_{pf}(G, 3) = \{\{v_1, v_4, v_6\}, \{v_1, v_4, v_8\}, \{v_2, v_3, v_6\}, \{v_2, v_3, v_8\}\}$.

$\implies d_{pf}(G, 3) =| D_{pf}(G, 3) |= 4$.

$D_{pf}(G, 4) = \{\{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_3, v_8\}, \{v_2, v_3, v_4, v_6\}, \{v_2, v_3, v_4, v_8\}\}$.

$D_{pf}(G, 5) = \{\{v_1, v_2, v_3, v_4, v_6\}, \{v_1, v_2, v_3, v_4, v_8\}\}$

$D_{pf}(G, 6) = \{\{v_1, v_4, v_5, v_6, v_7, v_8\}, \{v_2, v_3, v_5, v_6, v_7, v_8\}\}$

$D_{pf}(G, 7) = \{\{v_1, v_2, v_3, v_5, v_6, v_7, v_8\}, \{v_2, v_3, v_4, v_5, v_6, v_7, v_8\}\}$

$D_{pf}(G, 8) = \{\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}\}$

Hence, $d_{pf}(G, 4) =| D_{pf}(G, 4) |= 4$, $d_{pf}(G, 5) =| D_{pf}(G, 5) |= 2$,

$d_{pf}(G, 6) =| D_{pf}(G, 6) |= 2$, $d_{pf}(G, 7) =| D_p f(G, 7) |= 2$ and $d_{pf}(G, 8) =| D_{pf}(G, 8) |= 1$

The Perfect Domination Polynomial of G is given by

$$D_{pf}(G,x) = \sum_{i=3}^{8} d_{pf}(G,i)x^i = 4x^3 + 4x^4 + 2x^{5+}2x^6 + 2x^7 + x^8 \tag{2}$$

From (1) and (2) we get $D_{pf}(G,x) = D_{pf}(G_1,x)D_{pf}(G_2,x)$

Corollary 2.5. If G be a graph consisting of n components $G_1, G_2 \cdots, G_n$ then

$$D_{pf}(G,x) = D_{pf}(G_1,x)D_{pf}(G_2,x)\cdots D_{pf}(G_n,x)$$

The proof of Corollary 2.5 follows from Theorem 2.3.

Corollary 2.6. If $\overline{K_n}$ be the empty graph with n vertices then, $D_{pf}(\overline{K_n},x) = x^n$.

Proof. Since $\overline{K_n}$ is an empty graph with n vertices, it contains n components. With the help of Corollary 2.5, $D_{pf}(\overline{K_n},x) = x^n$. ■

Theorem 2.7. Let $K_{m,n}$ be a complete bipartite graph with $m+n$ vertices then, $D_{pf}(K_{m,n},x) = mnx^2 + x^{m+n}$ for $m,n > 1$.

Proof. Since G is a complete bipartite graph, the vertex set $V(G)$ can be partitioned into two disjoint subsets V_1 and V_2 of degree m and n respectively. Also, every vertex of V_1 is adjacent to n vertices of V_2 but not in V_1. Similarly every vertex of V_2 is adjacent to m vertices of V_1 but not in V_2. Hence, $d_{pf}(K_{m,n},(m+n)-1) = 0$ for all m,n and $d_{pf}(K_{m,n},m+n) = 1$. Therefore, $D_{pf}(G,x) = x^{m+n}$ for $m,n > 1$. ■

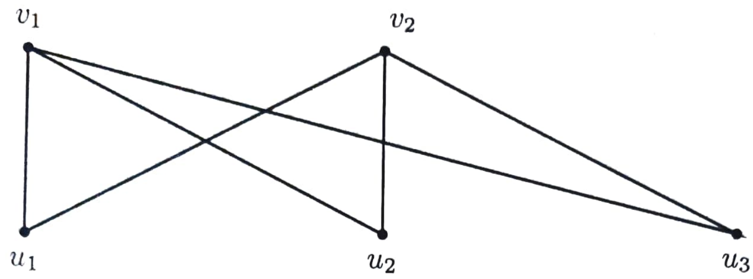

Figure 2.3. Graph in Example 2.8

Example 2.8. Consider the following complete bipartite Graph $K_{3,2}$ with five vertices as in Figure 2.3.

Now, we find the perfect domination polynomial of $K_{3,2}$. Note that, $\gamma_{pf}(K_{3,2}) = 2$. $D_{pf}(K_{3,2}, 2) = \{\{v_1, u_1\}, \{v_1, u_2\}, \{v_1, u_3\}, \{v_2, u_1\}, \{v_2, u_2\}, \{v_2, u_3\}\}$ Hence, $d_{pf}(K_{3,2}, 2)$ $= \mid D_{pf}(K_{3,2}, 2) \mid = 6$. From the Figure 2.3, we note that $D_{pf}(K_{3,2}, 3) = \phi$ and $D_{pf}(K_{3,2}, 4) = \phi$. Therefore, $d_{pf}(K_{3,2}, 3) = d_{pf}(k_{3,2}, 4) = 0$. $D_{pf}(k_{3,2}, 5) = \{\{v_1, v_2, v_3, v_4, v_5\}\}$ Hence, $d_{pf}(K_{3,2}, 5) = \mid D_{pf}(K_{3,2}, 5) \mid = 1$. Therefore, the perfect domination polynomial of $K_{3,2}$ is given by $D_{pf}(K_{3,2}, x) = \sum_{i=2}^{5} d_{pf}(K_{3,2}, i)x^i = 6x^2 + x^5$.

Theorem 2.9. If G is a complete bipartite graph $K_{1,n}$ then $D_{pf}(G, x) = x(1 + x)^n$.

Proof. Since the complete bipartite graph of the form $K_{1,n}$ is a star graph with $n + 1$ vertices [6] then we have, $D_{pf}(G, x) = x(1 + x)^n$. ■

Theorem 2.10 The degree of the perfect domination polynomial of any simple graph G is the order of the graph G.

Proof. Let G be a simple graph of order n, then the perfect domination number $\gamma_{pf} \leq n$.

Also, $V(G)$ is a perfect dominating set with cardinality n, then the perfect dominating sets of G with cardinality greater than n will be always empty. So, the degree of perfect domination polynomial is n, which is the order of the graph G. Therefore, degree of the perfect domination polynomial of any simple graph G is the order of G. ∎

Theorem 2.11. Let G be a simple graph and H be a subgraph of G. Then the degree of a perfect domination polynomial of H is less than or equal to the degree of a perfect domination polynomial of G.

Proof. As H is a subgraph of G, the order of H is always less than or equal to the order of G. By Theorem 2.10 the degree of perfect domination polynomial of H and G are the order of H and G respectively. Therefore, the degree of a perfect domination polynomial of H is less than or equal to the degree of a perfect domination polynomial of G. ∎

Theorem 2.12. In a graph G with n vertices

(i) If G is connected, then $d_{pf}(G, n) = 1$.

(ii) $d_{pf}(G, i) = 0$ iff $i < \gamma_{pf}(G)$ for $i > n$.

(iii) $d_{pf}(G, x)$ has no constant term

(iv) Zero is the root of $D_{pf}(G, x)$ with multiplicity $\gamma_{pf}(G)$.

(v) $D_{pf}(G, x)$ is a strictly increasing function on $[0, \infty)$.

Proof. (i) For any graph G, $V(G)$ is always a perfect dominating set of G, therefore there exists a perfect dominating set with cardinality n. Hence, $d_{pf}(G, n) = 1$.

(ii) The Proof follows from the definition of the perfect domination number.

(iii) Since, $\gamma_{pf}(G) \geq 1$, $D_{pf}(G,x)$ has no terms of degree zero. Hence, $D_{pf}(G,x)$ has no constant term.

(iv) If $D_{pf}(G,x) = 0$ then $x = 0$,. Hence zero is the root of $D_{pf}(G,x)$ since $D_{pf}(G,x)$ has no constant term. Also, the least power of x in $D_{pf}(G,x)$ is $\gamma_{pf}(G)$. Therefore, Zero is the root of $D_{pf}(G,x)$ with multiplicity $\gamma_{pf}(G)$.

(v) Since, $\frac{d(D_{pf}(G,x))}{dx} > 0$ on $(0,\infty)$, the perfect domination Polynomial is increasing in $[0,\infty)$. ■

3. Perfect domination polynomial of a double star graph

Definition 3.1. Let $B_{m,n}$ be a double star graph with $m+n+2$ vertices and $D_{pf}(B_{m,n}, i)$ be the family of perfect dominating sets of the double star graph $B_{m,n}$ with cardinality i then, $d_{pf}(B_{m,n}, i) = \mid D_{pf}(B_{m,n}, i) \mid$.

Lemma 3.2. $\gamma_{pf}(B_{m,n}) = 2$ for $m, n \in N$.

Proof. By the definition of double star graph, $B_{m,n}$ has $m+n+2$ vertices and $m+n+1$ edges. Usually, $B_{m,n}$ is constructed by joining the centre vertices of the two star graphs $K_{1,m}$ and $K_{1,n}$. But, $\gamma_{pf}(K_{1,m}) = \gamma_{pf}(K_{1,n}) = 1$ for $n \in N$. Hence, $\gamma_{pf}(B_{m,n}) = 2$ for $m, n \in N$. ■

Definition 3.3. Let $B_{m,n}$ be a double star graph with $m+n+2$ vertices. Then the

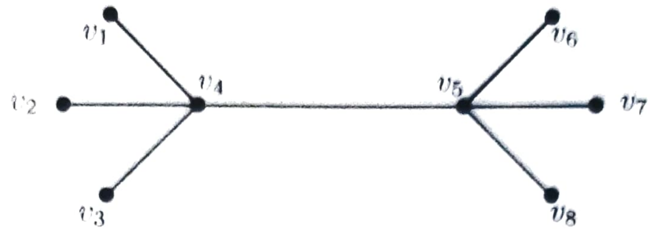

Figure 3.1. G_1 and G_2 of Example 3.4

perfect domination polynomial of the double star graph $B_{m,n}$ is given by $D_{pf}(B_{m,n}, x) = \sum_{i=2}^{m+n+2} d_{pf}(B_{m,n}, i)x^i$.

Example 3.4. Consider the following bi star graph $B_{3,3}$ in Figure 3.1.

As, $\gamma_{pf}(B_{n,n}) = 2$ for $n \in N$, the perfect dominating set of cardinality one is empty. We have, $d_{pf}(B_{3,3}, 1) = 0, d_{pf}(B_{3,3}, 2) = 1, d_{pf}(B_{3,3}, 3) = 6, d_{pf}(B_{3,3}, 4) = 15, d_{pf}(B_{3,3}, 5) = 20, d_{pf}(B_{3,3}, 6) = 15, d_{pf}(B_{3,3}, 7) = 6$ and $d_{pf}(B_{3,3}, 8) = 1$.

Therefore, the perfect domination polynomial of the bistar graph $B_{3,3}$ is given by

$$D_{pf}(B_{3,3}, x) = \sum_{i=2}^{8} d_{pf}(B_{3,3}, i)x^i$$

$$= d_{pf}(B_{3,3}, 2)x^2 + d_{pf}(B_{3,3}, 3)x^3 + d_{pf}(B_{3,3}, 4)x^4 + d_{pf}(B_{3,3}, 5)x^5 + d_{pf}(B_{3,3}, 6)x^6$$

$$+ d_{pf}(B_{3,3}, 7)x^7 + d_{pf}(B_{3,3}, 8)x^8 = x^2 + 6x^3 + 15x^4 + 20x^5 + 15x^6 + 6x^7 + x^8.$$

Lemma 3.5. For $i \geq 2, d_{pf}(B_{m,n}, i) = \binom{m+n}{i-2}$.

Proof. Let $B_{m,n}$ be the double star graph with $m + n + 2$ vertices and $m + n + 1$ edges. Note that, every perfect dominating set of the double star graph $B_{m,n}$ must contain two center vertices. Therefore, the perfect dominating set of the double star graph $B_{m,n}$ with

i vertices is obtained by choosing $i-2$ vertices from $m+n$ vertices. There are $\binom{m+n}{i-2}$ Possible ways. Hence, $d_{pf}(B_{m,n}, i) = \binom{m+n}{i-2}$. ■

Theorem 3.6 The Perfect domination polynomial of the double star graph $B_{m,n}$ is given by $D_{pf}(B_{m,n}, x) = x^2(1+x)^{m+n}$.

Proof. By the definition, the perfect domination polynomial of the double star graph is given by $D_{pf}(B_{m,n}, x) = \sum_{i=2}^{m+n+2} d_{pf}(B_{m,n}, i)x^i$. Using Lemma 3.5,

$$\begin{aligned} D_{pf}(B_{m,n}, x) &= \binom{m+n}{0}x^2 + \binom{m+n}{1}x^3 + \cdots + \binom{m+n}{m+n-2}x^{m+n} \\ &+ \binom{m+n}{m+n-1}x^{m+n+1} + \binom{m+n}{m+n}x^{m+n+2} \\ &= x^2[\binom{m+n}{0} + \binom{m+n}{1}x + \cdots + \binom{m+n}{m+n}x^{m+n}] \\ &= x^2(1+x)^{m+n}. \end{aligned}$$

■

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