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ON FUZZY INJECTIVE FUZZY G-MODULES

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Abstract: The notion of fuzzy G-module injectivity is introduced and analysed. It is proved that every finite dimensional quasi-injective G-module is the direct sum of quasy-injective G-submodules; and if the direct sum of fuzzy G-submodules on these G-modules is fuzzy quasi-injective, then the summands are also fuzzy quasi injective.

1.Introduction

The fuzzy set theory was introduced by L.A. Zadeh[1] in 1965 . There were several attempts to fuzzify various mathematical structures. The fuzzification of algebraic structures was initiated by Rosenfield[2]. He introduced the notions of fuzzy subgroupoids and fuzzy subgroups and derived some of their basic properties.

Eckmann and Schopf [3] introduced the notion of injectivity of modules. They also proved the existence of injective hull. A number of generalizations of this notion were defined and studied by several mathematicians. [*Egs*: pseudo-injective modules, ker-injective modules, π-injective module,etc.]. The author studied about fuzzy G-modules and fuzzy representation in[7]. As a continuation of work[7], here the concept fuzzy G-module injectivity is introduced and analysed.

2. Preliminaries

Throughout this paper G denotes a group. A vector space M over a field K is called a ***G-module*** if for every $g \in G$ and $m \in M$, $\exists$ a product (called the ***action of G on M***) g.m satisfying the following axioms:

(i) $m.I_G = m$, $\forall m \in M$(I_G being the identity element in G)
(ii) $m.(g.h) = (m.g).h$, $\forall m \in M; g,h \in G$; and
(iii) $(k_1m_1 + k_2m_2).g = k_1(m_1.g) + k_2(m_2.g), \forall k_1,k_2 \in K; m_1,m_2 \in M; g \in G$

What we have defined above may be rightly called the ***right action*** of G on M. In a similar way, we can define the ***left-action*** and ***left G-module***. In this paper, we shall restrict our attention to left G-modules. Similar results can be obtained for right G-modules also.

For G-modules M and M*, a mapping $\varphi: M \to M^*$ is said to be a ***G-module homomorphism*** if $\forall m,m_1,m_2 \in M; g \in G$ and $k_1,k_2 \in K$:

(i) $\varphi(k_1m_1+k_2m_2) = k_1.\varphi(m_1) + k_2.\varphi(m_2)$; and
(ii) $\varphi(g.m)=g.\varphi(m)$

All the terms and notations used in this paper are either standard or are explained as and when they appear.

2.1. Definition. A G-module M is ***injective*** if for any G-module M* and any G-submodule N of M*, every ***homomorphism*** from N into M can be extended to a homomorphism from M* into M.

2.2. Example. Let $G = C-\{0\}$, the multiplicative group of non-zero complex numbers.Let M =C, which is a vector space over C. Then M is a G-module w.r.t. trivial G-action. Also, except the zero G-sub module, no proper subset of C becomes a G-module. Further, if X is any G-module and N is any G -submodule of X, any homomorphism $\varphi: N \rightarrow M$ can be extended to a homomorphism from X to M. Therefore M is injective. □

2.3. Definition. Let M be a G-module. A G-module N containing M is called an ***injective hull*** of M if (i) N is injective (ii) There is no injective G-module L such that $M \subset L \subset N$. We denote N by E(M).

2.4. Remark. From the definition of injective hull, it is the minimal, proper G-module extension of the given G-module. If no such extention of M exits then E(M)=M.

2.5. Definition. Let M* and M be G-modules. Then M is ***M*-injective*** if for every G-submodule N of M*,any homomorphism $\varphi: N \rightarrow M$ can be extended to a homomorphism $\psi: M^* \rightarrow M$

2.6. Example. Let $M^* = R^n$, the n-dimensional vector space over R . Let $\{m_1, m_2, \ldots\ldots m_n\}$ be a basis for M* .Then $M^* = Rm_1 \oplus Rm_2 \oplus \ldots\ldots \oplus R_{m_n}$. Let M =R and G = any finite multiplicative subgroup of R. Then both M and M* are G-modules.Let X be any G-sub module of M* and $\varphi: X \rightarrow M$ be a homomorphism.(i) If $X = \{0\}$then $\varphi = 0$, then $\psi = 0 : M^* \rightarrow M$ extends φ. (ii). If $X = R_{m_j}$ $(1 \leq j \leq n)$. Then ψ: $M^* \rightarrow M$ defined by $\psi(c_1m_1 + \ldots\ldots c_jm_j \ldots\ldots + c_nm_n) = \varphi(c_jm_j)$ is a homomorphism and ψ extends φ.
(iii) $X = \bigoplus_{j=1}^{k} Rm_j$, $(k \leq n)$. Then,$\psi : M^* \rightarrow M$ defined by $\psi(c_1m_1 + \ldots\ldots + c_km_k + \ldots\ldots\ldots + c_nm_n) = \varphi(c_1m_1 + \ldots\ldots + c_km_k)$ extends φ. Therefore M is M*-injective. □

2.7. Proposition. *If M is a G-module and N is a G-submodule of M, then M/N is a G-module.*

Proof. Let $g \in G$ and $x+N \in M/N$
Define $g(x+N) = gx+N \in M/N$. Then M/N satisfies all the conditions of a G-module. □

2.8. Proposition. *Let M and M* be G-modules such that M is M*-injective. If N* is a G- submodule of M*, then M is N*-injective and Mis M* / N*-injective.*

Proof. $N^* \subseteq M^*$ and M is M*-injective implies M is N*-injective.Let X^1/N^* be a G-submodule of M^*/N^* and $\varphi : X^1/N^* \to M$ be a homomorphism. Let $\pi: M^* \to M^*/N^*$ be the projection map and $\pi^1 = \pi/X^1$. Then $\varphi \bullet \pi^1 : X^1 \to M$ is a homomorphism. Since M is M*-injective, $\exists$ an extension $\theta : M^* \to M$ homomorphism of $\varphi \bullet \pi^1$. Then $\theta(N^*) = \varphi \bullet \pi^1(N^*) = \varphi(\pi^1(N^*)) = \varphi(0) = 0$. Therefore Ker.$\pi$ is a G-submodule of Ker.θ. So $\exists$ a map $\psi : M^*/N^* \to M$ such that $\psi \bullet \pi = \theta$. Also for any $x \in X$, $\psi\,(X+N^*) = \psi\,(\pi(x)) = \theta(x) = (\varphi \bullet \pi^1)(x) = \varphi\,(x+N^*)$. Therefore ψ extends φ . Hence M is M*/N*-injective. □

2.9. Definition. A G-module M is ***Quasi-injective*** if M is M-injective.

2.10. Example. Let G = C-{0} and M = C which is a vector space over C Then M is a G- module. Also from example 2.2 ,M is injective. Hence M is X-injective, for any G- submodule X of M. In particular, let X =M, then M is M-injective. Therefore M is quasi-injective

2.11. Definition. Let M and M^1 be G-modules. Then M and M^1 are ***relatively injective*** if M is M^1-injective and M^1 is M-injective.

3.Injectivity and Quasi-injectivity of fuzzy G-modules

3.1. Definition. Let M and M^1 be G-modules. Let μ be any fuzzy G-module on M and v be any fuzzy G-module on M^1. Then μ is ***v-injective*** if (i) M is M^1-injective, and (ii) $v\,(m^1) \leq \mu(\psi(m^1))$, for every $\psi \in \mathrm{Hom}(M^1, M)$.

3.2. Example. Let G = (i) = {1, -i, v, -i} . Let M = C and M^1= Q(i) Then M and M^1 be G-modules. Define $\mu: M \to [0,1]$ by

$$\begin{aligned} \mu(x) &= 1, \quad \text{if } x = 0 \\ &= \tfrac{1}{2}, \text{ if } x \in C - Q(i) \\ &= 0, \quad \text{if } x \in Q(i) - \{0\} \end{aligned}$$

Then μ be a fuzzy G-module on M. Define $\nu : M^1 \to [0,1]$ by $\nu(x)=0$, $x \in M^1$. Then ν be a fuzzy G-module on M^1. Let X be any G-submodule of M^1. Then either $X = \{0\}$ or $X = M^1$. Let $\varphi : X \to M$ be any homomorphism (i) If $X=\{0\}$, then $\varphi = 0$, so $\psi = 0 : M^1 \to M$ extends φ. (ii) If $X= M^1$, then $\psi = \varphi$ extends φ. Therefore M is M^1- injective. Also $\nu(m^1) \le \mu(\psi(m^1))$, for every $\psi \in Hom(M^1,M)$ and $m^1 \in M^1$. Therefore is ν-injective. □

3.3. Proposition. *Let M be a G-module and N be a G-submodule of M. If M has a fuzzy G-module, then N and M/N has fuzzy G-modules.*

Proof. Let μ be a fuzzy G-module on M then $\nu = {}^{\mu}|_N$ is a fuzzy G-module on N. Define $\psi : M/N \to [0,1]$ by $\psi(x+N) = \mu(x)$, $x+N \in M/N$. Then ψ is a fuzzy G-module on M/N. □

3.4. Proposition. *Let M and M^l be G-modules such that M is finite dimensional and M is M^l- injective. Let $B =\{\beta_1, \beta_2, \ldots\ldots\ldots \beta_n\}$ be a basis for M. If $\nu(m^l) \le \min\limits_j \mu(\beta_j)$ for all $m^l \in M^l$, where μ and ν are fuzzy G-modules on M and M^l respectively, then μ is ν-injective.*

Proof. Let F be the scalar field of M and $\psi \in Hom(M^1, M)$. Since μ be a fuzzy G-module on M,we have

$$\mu(ax+by) \ge \min\{\mu(x), \mu(y)\}, \text{ where } x,y \in M,\ a,b \in K \qquad \textbf{(1)}$$

Let $m^1 \in M^1$. Then $\psi(m^1) \in M$, so $\psi(m^1) = a_1\beta_1+a_2\beta_2+\ldots\ldots\ldots a_n\beta_n$; $a_i \in K$, $\beta_i \in M$. Then $\mu(\psi(m^1)) \ge \min\limits_j \mu(\beta_j) \ge \nu(m^1)$ [from **(1)** and assumption] Therefore $\nu(m^1) \le \mu(\psi(m^1))$ for all $\psi \in Hom(M',M)$ and $m^1 \in M^1$. Therefore μ is ν -injective. □

3.5. Definition. Let μ and μ^1 be fuzzy G-modules on a G-module M. Then μ^1 ***exceeds*** μ if $\mu(m) \le \mu^1(m)$ for all $m \in M$.

3.6. Example. Let M= C and G = (i) .Then M be a G-module. Define $\mu, \mu^1 : M \to [0,1]$ by

$$\begin{aligned} \mu(x) &= \tfrac{1}{2}, \text{ if } x = 0 \\ &= \tfrac{1}{4}, \text{ if } x \ne 0 \text{ and} \end{aligned}$$

$$\mu^1(x) = 1, \text{ if } x = 0$$
$$= \tfrac{3}{4}, \text{ if } x \text{ is non-zero real}$$
$$= \tfrac{1}{2}, \text{ if } x \text{ is purely complex}$$

Then μ and μ^1 are fuzzy G-modules on M and $\mu(m) \leq \mu^1(m)$ for all $m \in M$. Therefore μ^1 exceeds μ. □

3.7. Proposition. *Let M and M' be G-modules , μ be a fuzzy G-module on M and v be a fuzzy G-module on M' such that μ is v-injective. If N^1 is a G-submodule of M^1 and v^1 be a fuzzy G-module on N^1 then μ is v^1-injective if v exceeds v^1 on N^1.*

Proof. Since M is M^1-injective and N^1 is a G-submodule of M^1, it follows from Proposition 2.8. that M is N^1-injective. Let $\psi \in Hom(N^1,M)$. Since M is M^1-injective, $\exists$ an extension homomorphism $\varphi: M \rightarrow M^1$ so that $\varphi/N^1=\psi$. Since μ is v -injective, $v(n') \leq \mu(\varphi(n')) = \mu(\psi(n^1))$, for all $n^1 \in N^1$. Given $v^1(n^1) \leq v(n^1)$. Therefore $v^1(n^1) \leq \mu(\psi(n^1))$ for all $\psi \in Hom(N^1,M)$ and hence μ is v^1 –injective. □

3.8. Remark. Let μ be a fuzzy G-module on M and let $r \in [0,1]$. Let $\mu_r: M \rightarrow [0,1]$ be defined by $\mu_r(m) = r.\ \mu(m)$, $\forall m \in M$. Then μ_r is also a fuzzy G-module on M and μ exceeds μ_r for any $r \in [0,1]$.

3.9. Proposition. *Let M and M^1 be G-modules and let the scalar field of M^1 be a subfield of C. Let μ be be a fuzzy G-module on M and v be a fuzzy G-module on M^1 such that μ is v-injective. Then for every $r \in [0,1]$ v_r is a fuzzy G-module on M^1 and μ is v_r-injective.*

Proof: Follows from the remark 3.8 and proposition 3.7. □

3.10. Proposition. *Let μ and v be fuzzy G-modules on G-modules M and M^1 respectively such that μ is v-injective.LetN^1 be a G-submodule of M^1 and $v^1 = {}^{v}|_N$. Define $\psi : M^1/N^1 \rightarrow [0,1]$ by $\psi(m^1 + N^1) = v(m^1)$, $m^1 \in M^1$. Then ψ is a fuzzy G-module on M^1/N^1 and μ is ψ-injective.*

Proof. It follows from proposition that ψ is a fuzzy G-module on M^1/N^1 and μ is v^1-injective. Since N^1 is a G-submodule of M^1, by proposition 2.8, M is M^1/N^1-injective. Let $\varphi \in Hom(M^1/N^1,M)$. Since M is M^1-injective, $\exists$ an extension $\varphi^l \in Hom(M^l,M)$. Since μ is v-injective and $\varphi^l \in Hom(M^l,M)$, $v(m^l) \leq \mu(\varphi^l(m^l))$. Now, $\mu(\varphi^l(m^l+N^l)) = \mu(\varphi^l(m^l)+0) = \mu(1.\varphi^l(m^l)+1.0) \geq \mu(\varphi^l(m^l)) \wedge \mu(0) = \mu(\varphi^l(m^l))$. Hence $\psi(m^l+N^l) = v(m^l) \leq \mu(\varphi^l(m^l)) \leq \mu(\varphi^l(m^l+N^l)) = \mu(\varphi(m^l+N^l))$ for all $\varphi \in Hom(M^l/N^l,M)$ and $m^l+N^l \in M^l/N^l$. Therefore μ is ψ-injective. □

3.11. Definition. Let M be a G-module and μ be a fuzzy G-module on M. Then μ is ***fuzzy quasi-injective*** if (i) M is quasi-injective, and (ii) $\mu(m) \leq \mu(\psi(m))$, for every $\psi \in Hom(M,M)$ and $m \in M$.

3.12. Proposition. *Let M be a G-module and $M = \bigoplus_i^n M_i$, where M_i's are G-submodules of M. If $v_i(1 \leq I \leq n)$ be fuzzy G-modules on M_i, then $v : M \to [0,1]$ defined by $v(m) = \wedge_i\{v_i(m_i)\}$, where $m = \sum_i^n m_i \in M$, is a fuzzy G-module on M.*

Proof. Straight forward. □

3.13. Definition. The fuzzy G-module v on $M = \bigoplus_i^n M_i$ in the preceding proposition with $v(0) = v_i(0)$ for all i, is called the ***direct sum*** of the fuzzy G-modules v_i and is denoted by $v = \bigoplus_{i=1}^n v_i$. Now, we state some theorems regarding these notions, omiting their proof.

3.14. Theorem. *Let G be any group and let M and M* be G-modules. Then M is M*-injective if and only if $\psi(M^*)$ is a G-module of M for any $\psi \in Hom(E(M^*), E(M))$.*

Corollory (1). *Let G be any group and M be a G-module. Then M is quasi-injective iff $\psi(M)$ is a G-submodule of M for every $\psi \in End(E(M))$* □

Corollory (2). *Let M and M^l are relatively injective G-modules and G, a subgroup of R. If $E(M) \approx E(M^l)$ then $M \approx M^l$. In fact, any isomorphism $E(M) \to E(M^l)$ restricts to an isomorphism $M \to M^l$. Further, M and M^l are quasi-injective.* □

3.15. Theorem. *Let M be a G-module such that $M = \bigoplus_i^n M_i$, where M_i's* *are G-submodules of M. Let v_i's be fuzzy G-modules on M_1's and let $v= \overset{n}{\underset{1}{\oplus}} v_i$. Let μ be any fuzzy G-module on M. Then μ is v-injective if and only if v is v_i-injective for all i.* □

3.16. Theorem. *Let M_1 and M_2 be G-submodules of a G-module M such that $M = M_1 \oplus M_2$. If M is quasi-injective, then M_i is M_j-injective for i,j=1,2. If $v_i{}^1$s are fuzzy G-modules on M_i (i =1,2) such that $v = v_1 \oplus v_2$ and v is quasi-injective ,then v_i is v_j-injective (i,j = 1,2)* □

Corollory. *Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ be a G-module,where M_i's are G-submodules of M. If M is quasi-injective then M_i is M_j-injective for $1 \leq i,j \leq n$. Also if v_is are fuzzy G-modules on Mi's such that $v = \overset{n}{\underset{i=1}{\oplus}} v_i$ and if v is quasi-injective then v_i is v_j-injective for $1 \leq i,j \leq n$.* □

3.17.Theorem: *Converse of the preceding theorem holds if $M \subseteq E(M_i)$ for i = 1,2* □

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