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ON CAYLEY GRAPHS OF SOLVABLE GROUPS

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Abstract: The Cayley graphs of solvable groups has been studied by Edward Dobson[4] and many others. In this paper the existence of Quotient graph in the Cayley graph of a solvable group is proved.

Key words: Solvable groups, Cayley graphs, Quotient graph

1. Preliminaries.

For all the basic definitions other than the ones given in this paper refer to references [1] and [2].

1.1. Definition. A composition series $\{H_i\}$ of a group G is a subnormal series $\{H_i\}$ of subgroup of G such that all the factor groups H_{i+1}/H_i are simple.

1.2. Definition. A group G is said to be solvable if it has a composition series $\{H_i\}$ such that all factor groups H_{i+1}/H_i are abelian.

1.3. Definition. The least n such that the n^{th} commutator subgroup $G^{(n)}=\{e\}$ the identity element of the group Gis called the derived length of the solvable group G.

1.4. Definition.: Cayley digraph of a group G with generating set S denoted by Cay(G,S) is defined as follows.

(i) each element of G is a vertex of Cay(G,S)

(ii) for x and y in G there is an arc from x to y if and only if xs=y for some s □S.

1.5. Remark. Cayley graphs are those graphs whose vertices may be identified with elements of groups and adjacency relation may be defined by subsets of the group. There are several ways to draw the digraph of a group given by a particular generating set.

1.6. Definition. A quotient graph Q of a graph G is a graph whose vertices are blocks of a partition of the vertices of G and where block B is adjacent to block C if some vertex in B is adjacent to some vertex in C with respect to the edge set of G. In other words, if G has edge set E and vertex set V and R is the equivalence relation induced by the partition, then the quotient graph has vertex set V/R and edge set $\{([u]_R, [v]_R) \mid (u, v) \in E(G)\}$

1.7. Remark. There are several ways to draw the digraph of a group given by particular generating set, but always connections are uniquely determined by generating set. Headless arrows joining two vertices x and y indicates that there is an arc from x to y and an arc from y to x. This occurs when the generating set contains both an element and its inverse.

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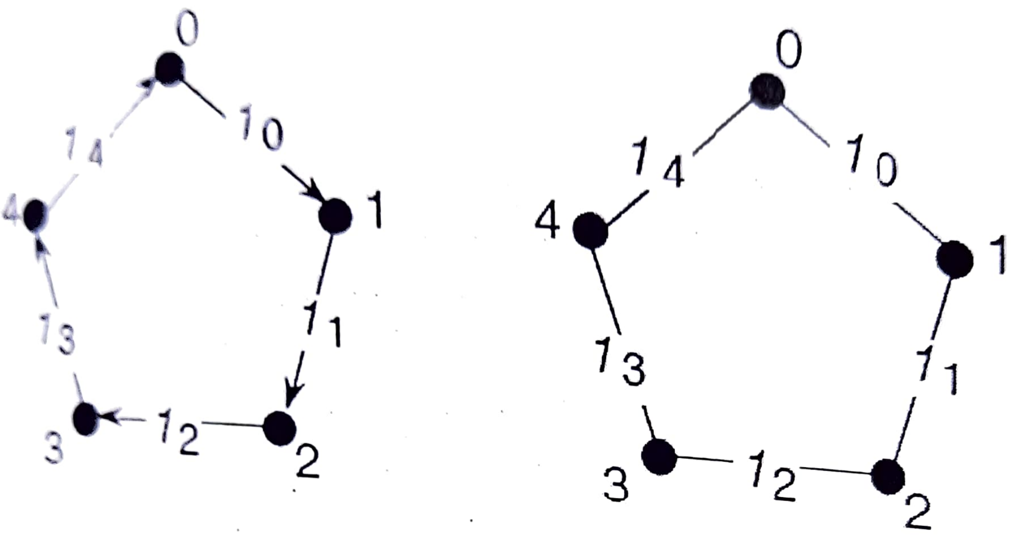
1.8. Example. Figure 1 shows the Cayley digraph for $Z5$ with generating set {1} and the corresponding Cayley graph.



Fig(1) : The Cayley digraph of Z_5 : $C(Z_5)$

2. The quotient in the Cayley graph of Solvablr Group

The following theorem proves that existence of the Quotient graph in the Cayley graph of a solvable group.

2.1. Theorem: The Cayley graph $X= Cay(G,S)$ of the solvable group of derived length n contains the quotient graph $\overline{X}$ of the corresponding factor group G/H_{n-1} where S is a generating set of G.

Proof: Let G be a solvable group and let S be a generating set of G such that e does not belong to S. Let $H_{n-1}= N$, the normal subgroup N which is normal subgroup of G in composition series $\{H_i\}$ such that $\frac{G}{N}$ is simple.

CASE 1: Assume that $N = G'$ the commutator subgroup and $G' \cap S = \emptyset$. For the case $N \cap S = \emptyset$ consider the quotient graph X obtained by first contracting every coset of G' to a single vertex. Since G' is a normal subgroup if some vertex of a coset $g\,G'$ is adjacent to d vertices of another coset $h\,G'$ then every vertex of $g\,G'$ is adjacent to d vertices of $h\,G'$ and vice-versa. Put an edge of multiplicity d between the vertices corresponding to the cosets $g\,G'$ and $h\,G'$ in $\overline{X}$. Note that since $G' \cap S = \emptyset$ no two adjacent vertices in X can be in the same coset of $\overline{X}$. The quotient graph $\overline{X}$ is thus formed.

CASE 2: Assume that $N \geq G'$, $N \cap S = \emptyset$. Then the argument is similar to the above case. For the case $N \cap S = \emptyset$ as N is a normal subgroup if some vertex of a coset gN is adjacent to d vertices of another coset hN then every vertex of gN is adjacent to d vertices of hN and vice-versa. Then put edge of multiplicity d between the vertices corresponding to the cosets gN and hN in $\overline{X}$. Also since $N \cap S = \emptyset$ no two adjacent vertices in X can be in the same coset of $\overline{X}$. The quotient graph $\overline{X}$ is thus formed.

CASE 3: Assume that $N \cap S \neq \emptyset$

Recall that N is a normal subgroup in G . Moreover G/N is abelian. Now consider the quotient graph X obtained by first contracting every coset of N to a single vertex. Since N is a normal subgroup if some vertex of a coset gN is adjacent to d vertices of another coset hN then every vertex of gN is adjacent to d vertices of hN and vice-versa. We can assume that if $a \in N \cap S$, Then aN is N itself. Hence number of edges in this case will be less than the number of edges in the case $N \cap S = \emptyset$. The quotient graph $\overline{X}$ is thus formed.

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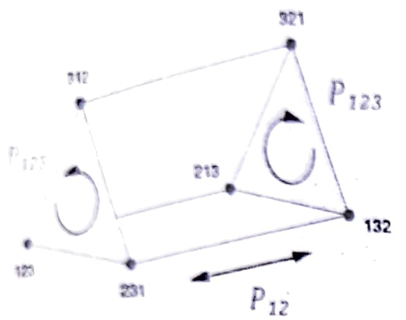
2.2. **Example.** Given below is an example of the Cayley graph of the group S_3 and S_3/A_3



Cayley graph of S_3 with generating set. {(123),(132),(12)}

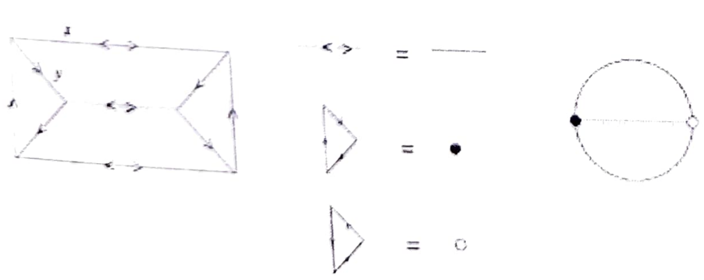
:Fig(2)

The red edges between the vertices indicate they are double edges. The blue edges are single edges with the direction given in the figure.

G = S3, and then A3={e,(123),(132)}.

Then S3/A3={A3,(12)A3} = {{e,(123),(132)},{(12),(23),(13)}}.

Cay(S_3/A_3, {A3,(12)A3}) is given in Fig (3) with explanation.



(a) (b) (c)

Fig(3)

In Fig(3) :(a) is Cayley graph if S_3, in (b) the line shows the edges are do edges, the dark vertex represents the block {e,(1 23),(132)} and the bu represents the block {(12),(23),(13)} (c) is the Cayley graph of S_3/A_3.

3. Conclusion

This result enriches the theory of Cayley graph of solvable groups and fur extension of this work is possible to find results on the length of composition series of solvable groups.

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