VOLUME IX; NUMBER 2 DECEMBER 2010

THE ALBERTIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

TAJOPAAM

A JOURNAL DEVOTED TO THE ENCOURAGEMENT OF

RESEARCH IN MATHEMATICS

Rev. Dr. A.O.Konnully Memorial Research Centre
Department of Mathematics
St. Albert's College, Ernakulam
Kochi-682018 , Kerala, India
Tel.: 0484-2394225; Fax:0484-2391245; E-mail: stalbertscol@sify.com

TAJOPAAM
Volume IX, Number 2
December 2010
Pages 214 - 231

LOCAL FUZZY IDEALS AND P- FUZZY IDEALS

Souriar Sebastian* and George Mathew**

**Department of Mathematics, St. Albert's college Ernakulam, Kochi- 682018, Kerala.*
***Department of Mathematics, B.C.M College, Kottayam-686001, Kerala, India*
e - mail: gmathew5616x@gmail.com

Abstract: In this paper, we study the algebra of local fuzzy ideals. We also introduce a special class of fuzzy ideals called P- fuzzy ideals where P is a prime ideal of a commutative ring and study the nature of a local P- fuzzy ideal in a Dedekind domain.

Key Words: Localization, Dedekind domains, fuzzy ideals, prime fuzzy ideals, local fuzzy ideals.

1.Introduction

It was Zadeh [17] who began the fuzzification of algebraic structures by introducing fuzzy sets. Rosenfeld [6] used this idea to define fuzzy groupoids and fuzzy groups. Liu [2-3] defined and studied fuzzy ideals in groups and rings. Some work done in this area by the authors are reported in [7-16]. The notion of local fuzzy ideals, fuzzy prime ideals and fuzzy primary ideals have been introduced and extensively studied in [5].

In this paper, we study the algebra of local fuzzy ideals. We also introduce a special class of fuzzy ideals called P- fuzzy ideals where P is a prime ideal of a commutative ring and study the nature of a local P- fuzzy ideal in a Dedekind domain.

2. Preliminaries

Throughout this paper, R denotes a commutative ring with unit element, unless otherwise stated. Algebraic terms and notations used here are either standard or as in [1], [4] and [18]. Given a universal set X, a function $\mu : X \longrightarrow [0, 1]$ is called a ***fuzzy subset*** of X. For a fuzzy subset μ of R and $t \in [0,1]$ the set $\mu_t = \{x \in R : \mu(x) \geq t\}$ is called the ***level set*** of μ corresponding to t. It may be recalled that a fuzzy subset μ of R is called a ***fuzzy ideal***[2] of R if (i) $\mu(x + y) \geq \mu(x) \wedge (y)$, (ii) $\mu(-x) = \mu(x)$

and (iii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$. Here $\wedge$ and $\vee$ denote the max (or sup) and min (or inf) operators respectively. We denote the set $\{x \in R : \mu(x) = \mu(0)\}$ by μ_* and the set $\{x \in R : \mu(x) > \mu(1)\}$ by $\mu_\#$.

2.1. Definition[5]. A fuzzy ideal μ of R is said to be a ***local fuzzy ideal*** if $\{x \in R : \mu(x) = \mu(1)\}$ = set of all units of R.

2.2. Proposition. A fuzzy ideal μ of R is a local fuzzy ideal if and only if $\mu(x) = \mu(1) \Rightarrow x$ is a unit of R.

Proof. Since μ is a fuzzy ideal $\mu(x) \geq \mu(1)$, $\forall\, x \in R$. Let x be a unit of R. Then there exists $y \in R$ such that $xy = 1$. $\therefore \mu(1) = \mu(xy) \geq \mu(x)$. It follows that $\mu(x) = \mu(1)$. Therefore μ is a local fuzzy ideal if $\mu(x) = \mu(1) \Rightarrow x$ is a unit of R.

2.3. Theorem [5]. Let μ be a non constant fuzzy ideal of R. Then μ is local $\Leftrightarrow \mu_\#$ is the unique maximal ideal of R.

2.4. Theorem [5]. A ring R is local $\Leftrightarrow$ R has a local fuzzy ideal.

2.5. Example. The ring $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ under addition and multiplication modulo 9 is local and $(3) = \{0, 3, 6\}$ is the unique maximal ideal. The fuzzy ideal μ defined by $(0) \longrightarrow 1$, $(3) - (0) \longrightarrow ¾$, $Z_9 - (3) \longrightarrow ½$ is a local fuzzy ideal as $\mu_\# = (3)$.

2.6. Example. If every element of a ring is either a nilpotent or a unit, then the set of all non units is the set of all nilpotent elements. The fuzzy ideal which assumes 1 at the nilpotent elements and 0 at unit elements is a local fuzzy ideal.

In the ring Z_9, the elements 0, 3, 6 are nilpotent elements while the elements 1, 2, 4, 5, 7, 8 are unit elements. Therefore the fuzzy ideal μ defined by

$$0, 3, 6 \longrightarrow 1; \qquad 1, 2, 4, 5, 7, 8 \longrightarrow 0$$

is a local fuzzy ideal.

2.7. Proposition. Suppose μ_1 and μ_2 are local fuzzy ideals of R, then $\mu_1 \cap \mu_2$ is also a local fuzzy ideal of R.

Proof. Suppose $(\mu_1 \cap \mu_2)(x) = (\mu_1 \cap \mu_2)(1)$. Then $\mu_1(x) \wedge \mu_2(x) = \mu_1(1) \wedge \mu_2(1)$. If $\mu_1(x) \wedge \mu_2(x) = \mu_1(x)$ and $\mu_1(1) \wedge \mu_2(1) = \mu_1(1)$, then since μ_1 is a local fuzzy ideal, x is a unit of R. The same holds if $\mu_1(x) \wedge \mu_2(x) = \mu_2(x)$ and $\mu_1(1) \wedge \mu_2(1) = \mu_2(1)$. If $\mu_1(x) \wedge \mu_2(x) = \mu_2(x)$ and $\mu_1(1) \wedge \mu_2(1) = \mu_1(1)$, then $\mu_2(x) = \mu_1(1) \leq \mu_2(1)$. But $\mu_2(x) \geq \mu_2(1)$. $\therefore$ $\mu_2(x) = \mu_2(1)$ so that x is a unit. The same holds if $\mu_1(x) \wedge \mu_2(x) = \mu_1(x)$ and $\mu_1(1) \wedge \mu_2(1) = \mu_2(1)$. Thus $(\mu_1 \cap \mu_2)$ is a local fuzzy ideal.

2.8. Remark. The above proposition is not true in the case of infinite intersection.

2.9. Example. Let R be a local Ring with unique maximal ideal M. Let μ_i, i $\in$ N, be defined by M $\longrightarrow$ 1/i, R – M $\longrightarrow$ 0. Then each μ_i is a local fuzzy ideal, however $\cap\ \mu_i = \{0\}$ is not a local fuzzy ideal.

2.10. Theorem. Let R be a local ring with unique maximal ideal M. Suppose μ and ν are local fuzzy ideals of R and $\mu(1) \neq \nu(1)$. Then μ ν is a local fuzzy ideal of R.

Proof. We have $(\mu\ \nu)(x) = \vee\{ \wedge_{i=1}^{n}\ \mu(y_i) \wedge \nu(z_i) : x = y_1z_1 + \ldots\ldots. + y_nz_n \}$.

In order to prove that μν is a local fuzzy ideal, it suffices to prove that $\mu\nu(x) = \mu\nu(1)$ for all $x \notin M$ and $\mu\ \nu(x) > \mu\ \nu(1)$ for all $x \in M$. Suppose $x \notin M$. Let

$$X = y_1z_1 + \ldots\ldots. + y_nz_n. \tag{1}$$

Then $y_jz_j \notin M$ for some j. Since M is an ideal, $y_j \notin M$ and $z_j \notin M$. Since μ and ν are local, $\mu(y_j) = \mu(1)$ (1) $\wedge\ \nu(1)$. ∴ $(\mu\nu)(x) \leq (\mu\nu)(1)$. But $(\mu\nu)(x) \geq (\mu\nu)(1)$ as μν is a fuzzy ideal. ∴ $(\mu\nu)(x) = (\mu\nu)(1)$ and that $(\mu\nu)(1) = \mu(1) \wedge \nu(1)$.

Suppose $x \in M$. Let $\nu(1) > \mu(1)$. We have $x = x.1$ and $\mu(x) > \mu(1)$ as μ is local. Now $\mu(x) \wedge \nu(1) > \mu(1) = \mu(1) \wedge \nu(1)$ ∴ $(\mu\nu)(x) \geq \mu(x) \wedge \nu(1) > \mu(1) \wedge \nu(1) = (\mu\nu)(1)$. ∴ $(\mu\nu)(x) > (\mu\nu)(1)$ if $\nu(1) > \mu(1)$. On the other hand, if $\mu(1) > \nu(1)$, then since $\nu(x) > \nu(1)$ as ν is local, $\mu(1) \wedge \nu(x) > \nu(1) = \mu(1) \wedge \nu(1)$. ∴$(\mu\nu)(x) \geq \mu(1) \wedge \nu(x) > \mu(1) \wedge \nu(1) = (\mu\nu)(1)$. Thus in either case $(\mu\nu)(x) > (\mu\nu)(1)$. ∴ μν is a local fuzzy ideal.

2.11. Remark. The conclusion in the above theorem does not hold if we delete the condition $\mu(1) \neq \nu(1)$.

2.12. Example. Let μ and ν be fuzzy ideals on Z_9 defined by

μ : $0, 3, 6 \longrightarrow 1$; $1,2,4,5,7,8 \longrightarrow ½$.

ν : $0 \longrightarrow 1$; $3, 6 \longrightarrow ¾$; $1, 2, 4, 5, 7, 8 \longrightarrow ½$

Then both μ and ν are local fuzzy ideals. We prove that $\mu\nu$ is not local. Consider an arbitrary representation of 3 in the form $3 = y_1z_1 + \ldots\ldots + y_nz_n$. If $y_iz_i \notin M$ $\forall$ i, then $y_i \notin M$ and $z_i \notin M$ $\forall$ i. $\therefore$ $\mu(y_i) = \mu(1)$ and $\nu(z_i) = \nu(1)$. Hence $\mu(y_i) \wedge \nu(z_i) = \mu(1) \wedge \nu(1)$.

$$\therefore \bigwedge_{i=1}^{n} \mu(y_i) \wedge \nu(z_i) = \mu(1) \wedge \nu(1).$$

On the other hand, if $y_jz_j \in M$ for some j, then either $y_jz_j = 3$ or $y_jz_j = 6$. In either case one among y_j and z_j not belong M. $\therefore$ $\mu(y_j) = \mu(1)$ or $\nu(z_j) = \nu(1)$. Hence $\mu(y_j) \wedge \nu(z_j) = ½ \wedge \nu(z_j) = ½$ since $\nu(x) \geq ½$ $\forall$ $x \in Z_9$. $\therefore$ $\mu(y_j) \wedge \nu(z_j) = \mu(1) = \nu(1) = \mu(1) \wedge \nu(1)$.

$$\therefore \bigwedge_{i=1}^{n} \mu(y_i) \wedge \nu(z_i) = \mu(1) \wedge \nu(1).$$

It follows that $(\mu\nu)(3) = \vee\{ \bigwedge_{i=1}^{n} \mu(y_i) \wedge \nu(z_i) : 3 = y_1z_1 + \ldots\ldots + y_nz_n \} = \mu(1) \wedge \nu(1) = (\mu\nu)(1)$. Since $3 \in M$, we can conclude that $\mu\nu$ is not a local fuzzy ideal.

2.13. Theorem. Let R be a local ring with unique maximal ideal M. If μ and ν are local fuzzy ideals of R and $\mu(0) = \nu(0)$, then $\mu + \nu$ is a local fuzzy ideal of R.

Proof. Since μ and ν are fuzzy ideals and $\mu + \nu$ is a fuzzy ideal. We first prove that $(\mu + \nu)(1) = \mu(1) \vee \nu(1)$. We have $(\mu + \nu)(1) = \vee \{\mu(y) \wedge \nu(z) : 1 = y + z\}$. Since $1 = 1 + 0$, it follows that $(\mu + \nu)(1) \geq \mu(1) \wedge \nu(0) = \mu(1) \wedge \mu(0) = \mu(1)$. i.e., $(\mu + \nu)(1) \geq \mu(1)$. Similarly since $1 = 0 + 1$, $(\mu + \nu)(1) \geq \mu(0) \wedge \nu(1) \geq \nu(0) \wedge \nu(1) = \nu(1)$. i.e., $(\mu + \nu)(1) \geq \nu(1)$. Thus

$$(\mu+\nu)(1) \geq \mu(1) \vee \nu(1). \qquad \textbf{(A)}$$

Now we prove that $(\mu + \nu)(1) \leq \mu(1) \vee \nu(1)$.

Case (i): $\mu(1) \geq \nu(1)$.

If $\mu(1) \geq \nu(1)$, then $\mu(1) \vee \nu(1) = \mu(1)$. Consider any two elements y and z in R. Let $\mu(y) \wedge \nu(z) > \mu(1)$. Then $\mu(y) > \mu(1)$ and $\nu(z) > \mu(1) \geq \nu(1)$. Since μ and ν are local fuzzy ideals, it follows that $y \in M$ and $z \in M$. Hence if y or z not in M, then $\mu(y) \wedge \nu(z) \leq \mu(1)$. Let $1 = y + z$. Then y or z not in M, for otherwise, M would not be maximal. Therefore $\mu(y) \wedge \nu(z) \leq \mu(1)$. Hence $\vee \{\mu(y) \wedge \nu(z) : 1 = y + z\} \leq \mu(1) = \mu(1) \vee \nu(1)$. i.e., $(\mu + \nu)(1) \leq \mu(1) \vee \nu(1)$.

Case (ii): $\nu(1) \geq \mu(1)$.

If $\nu(1) \geq \mu(1)$, then as in case (i), we can prove that if $1 = y + z$, then $\mu(y) \wedge \nu(z) \leq \nu(1)$.

$\therefore \vee \{\mu(y) \wedge \nu(z) : 1 = y + z\} \leq \nu(1) = \mu(1) \vee \nu(1)$. i.e., $(\mu + \nu)(1) \leq \mu(1) \vee \nu(1)$.

Thus in either case

$$(\mu+\nu)(1)\leq\mu(1)\vee\nu(1). \quad \text{(B)}$$

From (A) and (B) we can conclude that $(\mu+\nu)(1) = \mu(1) \vee \nu(1)$.

In order to prove that $(\mu+\nu)$ is a local fuzzy ideal, we have to prove that $(\mu+\nu)(x) > (\mu+\nu)(1)$ for $x \in M$ and $(\mu+\nu)(1) = (\mu+\nu)(1)$ for $x \notin M$. Let $x \in M$. If $\nu(1) \geq \mu(1)$. Then $\mu(1) \vee \nu(1) = \nu(1)$. Since ν is local, $\mu(0) = \nu(0) \geq \nu(x) > \nu(1)$ hence $\mu(0) \wedge \nu(x) = \nu(x) > \nu(1) = \mu(1) \vee \nu(1)$. Since $x = x + 0$, it follows that $\vee\{ \mu(y) \wedge \nu(z) : x = y + z \} > \mu(1) \vee \nu(1)$. On the other hand, if $\mu(1) \geq \nu(1)$, then since μ is local, $\nu(0) = \mu(0) \geq \mu(x) > \mu(1)$. Hence $\mu(x)\wedge\nu(0) = \mu(x) > \mu(1) = \mu(1) \vee \nu(1)$. Since $x = x + 0$, it follows that $\vee\{ \mu(y) \wedge \nu(z) : x = y + z \} > \mu(1) \vee \nu(1)$. Thus we have $(\mu+\nu)(x) > (\mu+\nu)(1)$. Let $x \notin M$. Since μ and ν are local, $\mu(x) = \mu(1)$ and $\nu(x) = \nu(1)$. If $x = y + z$, then either $y \notin M$ or $z \notin M$. Let $y \notin M$. Then $\mu(y) = \mu(1)$. $\therefore$ $\mu(y) \wedge \nu(z) = \mu(1) \wedge \nu(z)$. Now $\mu(1) \vee \nu(1) \geq \mu(1) \geq \mu(1) \wedge \nu(z) = \mu(y) \wedge \nu(z)$. i.e., $\mu(y) \wedge \nu(z) \leq \mu(1) \vee \nu(1)$. Similarly, if $z \notin M$, then also $\mu(y) \wedge \nu(z) \leq \mu(1) \vee \nu(1)$. Thus $\mu(y) \wedge \nu(z) \leq \mu(1) \vee \nu(1)$ for all $x = y + z$. Also $x = x + 0$ and $\mu(x) \wedge \nu(0) = \mu(1) \wedge \mu(0) = \mu(1)$. At the same time $x = 0 + x$ and $\mu(0) \wedge \nu(x) = \nu(0) \wedge \nu(1) = \nu(1)$. It follows that

$$\vee\{ \mu(y) \wedge \nu(z) : x = y + z \} = \mu(1) \vee \nu(1).$$

i.e., $(\mu+\nu)(x) = (\mu+\nu)(1)$. This proves that $\mu + \nu$ is a local fuzzy ideal

2.14. Remark. The above theorem fails if we delete the condition $\mu(0) = \nu(0)$.

2.15. Example. In the ring $Z_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ under addition and multiplication modulo 9, the following fuzzy ideals are local.

$$\mu: \quad 0, 3, 6 \longrightarrow 0.4; \quad 1, 2, 4, 5, 7, 8 \longrightarrow 0.3\,.$$

$$v: \quad 0 \longrightarrow 0.8; \quad 3, 6 \longrightarrow 0.7; \quad 1, 2, 4, 5, 7, 8 \longrightarrow 0.5.$$

Here $\mu(0) \neq v(0)$. $\mu + v$ is the constant fuzzy ideal

$$0, 1, 2, 3, 4, 5, 6, 7, 8 \longrightarrow 0.4.$$

It is not a local fuzzy ideal

2.16. Example. In the above example if we take $\mu(0) = 0.8$, then $\mu(0) = v(0)$ and the corresponding fuzzy ideal $\mu + v$ is defined by

$$0 \longrightarrow 0.8; \quad 3, 6 \longrightarrow 0.7; \quad 1, 2, 4, 5, 7, 8 \longrightarrow 0.5$$

That is $\mu + v = v$, hence $\mu + v$ is a local fuzzy ideal.

2.17. Theorem. Let R be a local ring having the unique maximal ideal M and μ be a local fuzzy ideal of R. If I is any ideal of R, then the fuzzy ideal ε of R/I defined by

$$\varepsilon(x + I) = \vee\{\, \mu(x + y) : y \in I \,\} \text{ is a local fuzzy ideal of } R/I\,.$$

Proof. Since R is a local ring with unique maximal ideal M, R/I is a local ring with unique maximal ideal M/I. To prove that ε is a local fuzzy ideal, we need to prove that if $x + I \in M/I$, then $\varepsilon(x + I) > \varepsilon(1 + I)$ and if $x + I \notin$

M/I, then $\varepsilon(x + I) = \varepsilon(1 + I)$. Let $x + I \in M/I$. Then $x \in M$. $\therefore \mu(x) > \mu(1)$. Since M is the only maximal ideal of R ana every ideal is contained in a maximal ideal, $I \subset M$. $\therefore y \in I \Rightarrow y \in M$, hence $x + y \in M$. $\therefore \mu(x + y) > \mu(1)$. Hence $\vee\{ \mu(x + y) : y \in I \} > \mu(1)$. Again, $\varepsilon(1 + I) = \vee\{ \mu(1+ y) : y \in I\}$. Since $1 \notin M$, for $y \in I$, $1 + y \notin M$. $\therefore \mu(1 + y) = \mu(1)$. Hence $\varepsilon(1 + I) = \mu(1)$. It follows that $\varepsilon(x + I) > \varepsilon(1 + I)$. Let $x + I \notin M/I$. Then $x \notin M$. If $y \in I$, then $y \in M$, hence $x + y \notin M$. $\therefore \mu(x + y) = \mu(1)$. $\therefore \varepsilon(x + I) = \vee\{ \mu(x + y) : y \in I \} = \mu(1)$. i.e., $\varepsilon(x + I) = \varepsilon(1 + I)$. Thus μ is a local fuzzy ideal of R/I.

2.18. Theorem. Let R and S be local rings and μ a local fuzzy ideal of R. If $f : R \longrightarrow S$ is a homomorphism of R onto S, $f(\mu)$ is a local fuzzy ideal of S.

Proof. Let M be the unique maximal ideal of R. Then $f(M)$ is the unique maximal ideal of S. Since μ is a local fuzzy ideal of R, $\mu(x) > \mu(1)$ for $x \in M$ and $\mu(x) = \mu(1)$ for $x \notin M$. Suppose $f(x) = 1$. Then $x \notin M$, for, otherwise $1 \in f(M)$ which is not possible since $f(M)$ is maximal. $\therefore \mu(x) = \mu(1)$,hence $f(\mu)(1) = \vee\{\mu(x) : f(x) = 1\} = \mu(1)$.

Let $y \in f(M)$. Then $y = f(x_0)$ for some $x_0 \in M$. $f(\mu)(y) = \vee\{\mu(x) : f(x) = y\} \geq \mu(x_0) > \mu(1) = f(\mu)(1)$. i.e., $f(\mu)(y) > f(\mu)(1)$.

Let $y \notin f(M)$. If $y = f(x)$, then $x \notin M$ so that $\mu(x) = \mu(1)$. $\therefore f(\mu)(y) = \vee\{\mu(x) : f(x) = y\} = \mu(1) = f(\mu)(1)$. i.e., $f(\mu)(y) = f(\mu)(1)$. Thus $f(\mu)$ is a local fuzzy ideal of S.

2.19.Theorem. Let R and S be local rings and f be an isomorphism of R on to S. If v is a local fuzzy ideal of S, then $f^{-1}(v)$ is a local fuzzy ideal of R.

Proof. Let M and M' be the unique maximal ideals of R and S respectively. Since f is on to, f(M) = M'. Let $x \in M$. Then $f^{-1}(v)(x) = v(f(x)) > v(1)$, since $f(x) \in f(M) = M'$ and v is local. Also $f^{-1}(v)(1) = v(f(1))$. Since f is one to one and $f(1) = 1 \notin M'$, $v(f(1)) = v(1)$. $\therefore f^{-1}(v)(x) > f^{-1}(v)(1)$. Let $x \notin M$. Then $f(x) \notin f(M) = M'$. $\therefore$ $v(f(x)) = v(1) = v(f(1))$. $\therefore$ $f^{-1}(v)(x) = v(f(x)) = v(f(1)) = f^{-1}(v)(1)$. $\therefore$ $f^{-1}(v)$ is a local fuzzy ideal of R.

3. P- Fuzzy ideals

Recall that if P is a prime ideal of R, and S = R – P, then the ring of quotients $S^{-1}R$ is called the ***localization*** of R at P denoted by R_P.

3.1. Proposition . Let R be a ring and P be a prime ideal of R. Let R_P be the localization of R at P. If μ is a fuzzy ideal of R, then the following conditions are equivalent:

(i) $r_1/s_1, r_2/s_2 \in R_P$ and $r_1/s_1 = r_2/s_2 \Rightarrow \mu(r_1) = \mu(r_2)$.

(ii) $r \in R$ and $s \in R - P \Rightarrow \mu(rs) = \mu(r)$

Proof. Assume condition (i). Let $r \in R$ and $s \in R - P = S$. Consider any $s \in S$. Then $r/s_1 \in R_P$ and $r/s_1 = rs/s_1s$. By hypothesis, $\mu(rs) = \mu(r)$. Conversely assume condition (ii). Let r_1/s_1 , $r_2/s_2 \in R_P$. If $r_1/s_1 = r_2/s_2$, then $r_1s_2 = r_2s_1$.

Therefore $\mu(r_1s_2) = \mu(r_2s_1)$. By condition (ii), this implies $\mu(r_1) = \mu(r_2)$. Thus conditions (i) and (ii) are equivalent.

3.2. Definition. Let R be a ring and P be a prime ideal of R. Then a fuzzy ideal μ of R is said to be a ***P – fuzzy ideal*** if it satisfies any of the above two conditions.

3.3. Example. Consider the ring of integers Z. The ideal P = (2) is a prime ideal of Z. Let μ be defined by

$$(8) \longrightarrow 1,\ (4)-(8) \longrightarrow \tfrac{1}{2},\ (2)-(4) \longrightarrow \tfrac{1}{3},\ Z-(2) \longrightarrow \tfrac{1}{4}.$$

Then μ is a fuzzy ideal on Z. Clearly $\mu(rs) = \mu(r)$ for all $r \in Z$ and $s \in Z-(2)$. Therefore μ is a P- fuzzy ideal of Z.

3.4. Example. In the ring of integers Z, consider the prime ideal P = (5). The fuzzy ideal μ defined by

$$(10) \longrightarrow a_1,\ (5)-(10) \longrightarrow a_2,\ Z-(5) \longrightarrow a_3 \text{ where } 1 \geq a_1 > a_2 > a_3 \geq 0$$

is not a P-fuzzy ideal because $2 \in Z-(5)$, but $\mu(10) = \mu(5.2) = a_2 \neq a_1 = \mu(5)$.

3.5. Proposition. Let R be a ring and P be a prime ideal of R. If μ is a P-fuzzy ideal of R, then every level set of μ not equal to R is a subset of P.

Proof. Let μ_t be a level set of μ and $\mu_t \neq R$. Let $x \in \mu_t$. If $x \notin P$, then $x \in R - P = S$. Since $\mu_t \neq R$, we can choose $y \in R$ but $y \notin \mu_t$. $\therefore$ $\mu(y) < t$. Since μ is a P-fuzzy ideal and $x \in S$, $\mu(xy) = \mu(y)$. $\therefore$ $\mu(xy) < t$, hence $xy \notin \mu_t$. But μ_t is an ideal and $x \in \mu_t$, therefore $xy \in \mu_t$. This is a contradiction. Hence $x \in P$. Thus every level set not equal to R is a subset of P.

4. P- Fuzzy ideals and Local fuzzy ideals of Dedekind domains

Recall that an integral domain R is a ***Dedekind domain*** if R satisfies any of the following equivalent conditions:

(i) If A and B are any two ideals of R, then $A \subseteq B \Rightarrow A / B$

(ii) R is Noetherian, integrally closed and every prime ideal of R is maximal.

(iii) R is Noetherian and R_P is a discrete valuation ring for all prime ideals P of R.

4.1. Proposition[4]. If R is a Dedekind domain, then every primary ideal of R is a power of a prime ideal.

4.2. Proposition. Let R be a Dedekind domain and P be a prime ideal of R. Then μ is a P-fuzzy ideal of R, if and only if every non-zero level set of μ is of the form P^r for some non-negative integer r.

Proof. Let μ_t be a level set of μ. If $\mu_t = R$, then $\mu_t = P^0$. Suppose $\mu_t \subsetneq R$. Then by proposition 3.7, $\mu_t \subseteq P$. Let μ_t be a proper subset of P. Since R is a Dedekind domain, $\mu_t = CP$ for some ideal C. We prove that $C \subseteq P$. If not, there exists $x \in C$ but $x \notin P$. But then $x \in R - P = S$. Let $y \in P$. Then $xy \in CP = \mu_t$. $\therefore \mu(xy) \geq t$. Since μ is a P- fuzzy ideal $\mu(xy) = \mu(y) \geq t$, hence $y \in \mu_t$. $\therefore P \subseteq \mu_t$. This is a contradiction since μ_t is a proper subset of P. $\therefore C \subseteq P$.

Again, since R is a Dedekind domain, $C = C_1P$ for some ideal C_1. Now $\mu_t = C_1P^2$. If $C_1 \not\subseteq P$, then there exist $z \in C_1$ but $z \notin P$. But then $z \in S$. Let $u \in P^2$. Then $zu \in C_1P^2 = \mu_t$. $\therefore \mu(zu) \geq t$. Since μ is a P-fuzzy ideal, $\mu(zu) = \mu(u) \geq t$, hence $u \in \mu_t$. $\therefore P^2 \subseteq \mu_t$. Since $\mu_t \subsetneq P$, $P^2 \subseteq \mu_t \subsetneq P$. It follows that $\mu_t = P^2$. On the other hand, if $C_1 \subseteq P$, then $C_1 = C_2P$. But then $\mu_t = C_2P^3$. Again if $C_2 \not\subseteq P$ then as proved above, we get $\mu_t = P^3$. Let P^r be the highest power of P contained in μ_t. Then $\mu_t = UP^r$ where $U \not\subseteq P$. As proved above, we get $\mu_t = P^r$. Thus every level set of μ is a power of P.

Conversely suppose that every non-zero level set of μ is a power of P. Since R is a Dedekind domain, every prime ideal of R is maximal. Let $0 \neq r \in R$ and $s \in S$. Suppose $rs \in P^{k+1}$. We prove that $r \in P^{k+1}$. Assume the contrary that $r \notin P^{k+1}$, but $r \in P^k$. Then $r \in P^{k+1} - P^k$. Since P maximal and $s \notin P$, we have $R = P + (s)$. $\therefore 1 = \alpha - xs$ where $\alpha \in P$ and $x \in R$. $\therefore \alpha - 1 = xs$, hence

$$\alpha r - r = xrs \tag{1}$$

Since $\alpha \in P$ and $r \in P^k$, $\alpha r \in P^{k+1}$. Now $\alpha r - r \in P^k - P^{k+1}$, for, if $\alpha r - r \in P^{k+1}$, then $\alpha r - r = y \in P^{k+1}$, but then $r = \alpha r - y \in P^{k+1}$ which is not possible. Thus $\alpha r - r \in P^k - P^{k+1}$. Therefore From(1), $xrs \in P^k - P^{k+1}$. But since $rs \in P^{k+1}$, $xrs \in P^{k+1}$. This is a contradiction. Hence $r \in P^{k+1}$. Thus $rs \in P^{k+1} \Rightarrow r \in P^{k+1}$. But $r \in P^{k+1} \Rightarrow rs \in P^{k+1}$. Thus $rs \in P^{k+1} \Leftrightarrow r \in P^{k+1}$.

Consider $\bigcap_i P^i$. Let $rs \notin \bigcap_i P^i$. Then $rs \notin P^i$ for some i. Let j be the lowest power such that $rs \notin P^j$. Then $rs \in P^{j-1} - P^j$. But then $r \in P^{j-1} - P^j$. Since the level sets of μ are all powers of P, it follows that $\mu(rs) = \mu(r)$. On the other hand, if $rs \in \bigcap_i P^i$, then since R is Noetherian, $rs = 0$. But then $r = 0$, since $s \neq 0$. $\therefore \mu(rs) = \mu(r)$. Thus μ is a P- fuzzy ideal.

4.3. Theorem. Let R be a Dedekind domain. If R possesses a local P- fuzzy ideal μ for some prime ideal P, then R is a local ring with unique maximal ideal P and every level set of μ is a power of P.

Proof. Suppose R possesses a local P- fuzzy ideal μ. Then we have

(i) $\mu(rs) = \mu(r)\ \forall\ r \in R$ and $s \in S = R - P$.

(ii) $\mu(r) = \mu(1) \Rightarrow r$ is a unit of R.

By Theorem 2.3 and Theorem 2.4, R is a local ring and the set of non units M is the unique maximal ideal of R. We prove that M = P. Let $x \in M$. Then x is a non unit. By (ii), $\mu(x) > \mu(1)$. If $x \notin P$, then $x \in S = R - P$. By condition (i), $\mu(x) = \mu(1.x) = \mu(1)$ which is not possible. $\therefore x \in P$. Hence $M \subseteq P$. Since M is maximal it follows that $M = P$. Finally, since R is a Dedekind domain, by proposition 4.2, every level set of μ is a power of P.

5. References

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