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INTERCONVERSIONS OF CRISP, FUZZY AND INTUITIONISTIC FUZZY SETS

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Abstract: Set theory was introduced by George Cantor in the 19th century and this was further developed into Fuzzy Set theory by L.A. Zadeh in the 1960's, allowing partial membership in a set. It was further generalized into Intuitionistic Fuzzy Sets by K.T. Atanassov in the 1980's, introducing an aspect of indeterminacy. In this paper we discuss methods for inter-conversions of crisp, fuzzy and intuitionistic fuzzy sets.

Key Words:Fuzzy Set,Intuitionistic Fuzzy Set, Membership Function, Non-membership Function, Hesitancy Grade, Indeterminacy, Fuzzification, De-fuzzification, I-fuzzification, Restricted Scalar Multiplication, α-multiplication, (α,β)-multiplication

1. Introduction

The theory of sets was formalised by George Cantor in the 19th century. His concept of sets, now described as classical sets or crisp sets or ordinary sets, has become fundamental to all branches of Mathematics.

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Zadeh [9] introduced the concept of a Fuzzy Set (FS), in 1964, as a generalisation of the ordinary set by involving a membership function in place of the characteristic function.

In 1982, Atanassov [2,3] further generalised this into intuitionistic fuzzy sets (IFS) by introducing a non-membership value, along with membership value. This accommodates an aspect of indeterminacy regarding membership and non-membership of an element in a given set.

When there are several types of sets, it is natural to ask whether one can be converted to the other. In this paper we address this problem and review some of the existing methods for the inter-conversions of crisp, fuzzy and intuitionistic fuzzy sets.

2. Preliminary Concepts

It is not easy to define the concept of a set and various approaches have been made to develop set theory formally, each with its own merits and demerits. We assume that 'set' and 'membership' are concepts intuitively known. However, the following observations are made to clarify the concepts.

2.1 Definition. The *universe of discourse* is the collection of all possible values for an input to a system. It may be considered as the collection of all elements for which the discussion is meaningful.

2.2. Definition. A *set* (or, *crisp set*) is a well defined collection of distinct objects, which are called *elements* or *members* of the set.

2.3. Remark. A crisp set partitions the universal set into two subsets – *members* and *non members*. In other words, a crisp set has a sharp boundary and every element is either in the set or outside of it. However, life situations are not always crisp and may involve vagueness. In such cases, fuzzy set theory is more advantageous.

2.4. Definition [7]. Let X be a (crisp) set. Then a *fuzzy set* A on X is defined by $A = \{(x, \mu_A(x)) / x \in X\}$.

Here μ_A is called the *membership function* and $\mu_A(x)$ is the *degree of membership* of x in the fuzzy set A.

2.5. Notation. The membership value $\mu_A(x)$ depends only on the set A and the element x. So we may denote it by A(x).

2.6. Definition [3]. An *Intuitionistic Fuzzy Set* (IFS) A in a universe X is an object $A=\{<x, A^+(x), A^-(x)>/x\in X\}$ with $0 \le A^+(x) \le 1$, $0 \le A^-(x) \le 1$ and $A^+(x) + A^-(x) \le 1, \forall x \in X$.

The functions $A^+: X \to [0, 1]$ and $A^- : X \to [0, 1]$ are respectively called the membership function and non membership function.

2.7. Definition [3]. Let A be an IFS on X and let $x\in X$. Then $A^0(x) = 1 - (A^+(x) + A^-(x))$ is called the *hesitancy grade* or simply, the *hesitancy* of x. It is the degree of uncertainty of x as a member of the set A. The hesitancy may also be called *indeterminacy*.

3. Conversion of Crisp Sets to Fuzzy Sets

3.1 Fuzzification. By fuzzification, we mean the process of converting an object into a fuzzy object. We may distinguish between two types of fuzzifications – fuzzification of crisp sets and fuzzification of IFSs. A crisp set may be fuzzified by attaching a membership grade to each element of the set. An IFS may be fuzzified by removing the hesitancy part. In this section, we consider the conversion of crisp sets to fuzzy sets.

3.2. Definition [5]. Let A be a fuzzy set on X and let $\alpha \in (0, 1]$. Then by αA, we mean a fuzzy set on X, given by $(\alpha A)(x) = \alpha(A(x))$, for every $x\in X$. This process of associating another fuzzy set with a given fuzzy set is called *restricted scalar multiplication.*

3.3. Remarks.

i. In general, $\alpha A \subseteq A$.

ii. αA is defined for crisp sets also. When A is a crisp set and $\alpha \neq 1$, αA is a (non-crisp) FS.

3.4. Example. Let $X = \{ x_1, x_2, x_3\}$, $A = \{<x_1, 0.5>, <x_2, 0.6>, <x_3, 0.4>\}$ and $\alpha = 1/2$. Then $\alpha A = \{<x_1, 0.25>, <x_2, 0.3>, <x_3, 0.2>\}$

3.5. Properties [5]. Let A and B be fuzzy sets on the same universe X and let α, $\beta \in (0, 1]$. Then,

i. $\alpha(A \cup B) = \alpha A \cup \alpha B$

ii. $\alpha(A \cap B) = \alpha A \cap \alpha B$

iii. $(\alpha\beta)A = \alpha(\beta A)$

iv. $\alpha A = A$ if and only if $\alpha = 1$.

3.6. .RSM Fuzzifier. The restricted scalar multiplication produces a fuzzy set from a given fuzzy set. However, it can be used to produce a fuzzy set from a given crisp set also. Hence, RSM is a fuzzification process.

3.7. Example. Let $A = \{x_1, x_2, x_3\}$ be a crisp set. It may be written in the form of a FS as

$A = \{<x_1, 1.0>, <x_2, 1.0>, <x_3, 1.0>\}$. Take $\alpha = 0.6$.

Then, $\alpha A = \{<x_1, 0.6>, <x_2, 0.6>, <x_3, 0.6>\}$.

Here, although the original set A is a crisp set, αA is a FS.

4. Conversion of Fuzzy Sets to Intuitionistic Fuzzy Sets

We extend the above method to produce IFSs from given FSs. The process of converting an object into an IFS may be called *i-fuzzification*. We consider two methods of i-fuzzification – *restricted scalar multiplication by* α or, *α-multiplication* and *restricted scalar multiplication by* (α,β) or, *(α,β)-multiplication*. Both are methods to produce an IFS associated with a given IFS. But, they may be conveniently used to construct IFSs from given FSs as is obvious from the following definitions and examples.

4.1. Definition [8]. Let $A = \{<x, A^+(x), A^-(x)> / x \in X\}$, be an IFS on X and let $\alpha \in (0, 1]$. Then by $M_\alpha(A)$, we mean an IFS on X, given by $M_\alpha(A) = \{<x, \alpha A^+(x), \alpha A^-(x)> / x \in X\}$. This process is called *restricted scalar multiplication by* α, or *α-multiplication.*

4.2.Example. Let $A = \{<x_1, 0.5, 0.1>, <x_2, 0.4, 0.2>, <x_3, 0.6, 0.2>\}$ and let $\alpha = 0.5$. Then, $M_\alpha(A) = \{<x_1, 0.25, 0.05>, <x_2, 0.2, 0.1>, <x_3, 0.3, 0.1>\}$.

4.3. I-fuzzification by α-multiplication. The following example illustrates the use of α-multiplication as a method for i-fuzzification.

Let A be a FS given by $A = \{<x_1, 0.5>, <x_2, 0.6>, <x_3, 0.4>\}$. We can write the FS A in IFS format as $A = \{<x_1, 0.5, 0.5>, <x_2, 0.6, 0.4>, <x_3, 0.4, 0.6>\}$.

Then $M_\alpha(A) = \{<x_1, 0.25, 0.25>, <x_2, 0.3, 0.2>, <x_3, 0.2, 0.3>\}$, which is obviously an IFS

4.4,Definition [8]. Let $A = \{<x, A^+(x), A^-(x)>/x \in X\}$, be an IFS on X and let $\alpha, \beta \in (0, 1]$. Then by $M_\alpha^\beta(A)$, we mean an IFS on X, given by $M_\alpha^\beta(A) = \{<x, \alpha A^+(x), \beta A^-(x)> / x \in X\}$. This process is called *restricted scalar multiplication by* (α, β), or*(α,β)-multiplication.*

4.5. Example. Let $A = \{<x_1, 0.5, 0.2>, <x_2, 0.8, 0.1>, <x_3, 0.4, 0.4>\}$, $\alpha = 0.5$ and $\beta = 0.4$. Then $M_\alpha^\beta(A) = \{<x_1, 0.25, 0.08>, <x_2, 0.4, 0.04>, <x_3, 0.2, 0.16>\}$.

4.6. I-fuzzification by (α,β)-multiplication. (α,β)-multiplication can also be used for i-fuzzification. Again, we illustrate the method by an example.

Let A be a fuzzy set given by A = {<x_1, 0.6>, <x_2, 0.8>, <x_3, 0.4>}. Let $\alpha = 0.5$ and $\beta = 0.4$. First we write A in the IFS format as A = {<x_1, 0.6, 0.4>, <x_2, 0.8, 0.2>, <x_3, 0.4, 0.6>}. Then, $M_\alpha^\beta(A)$= {<x_1, 0.3, 0.16>, <x_2, 0.4, 0.08>, <x_3, 0.2, 0.24>}.

4.7 Remarks. From the above definition, we observe the following:

i. If $\alpha \neq 1$, or $\beta \neq 1$ then, $M_\alpha^\beta(A)$ produces an associated IFS from a given IFS A.

ii. When A is FS, we can write it as an IFS in the form A = {<x, $A^+(x)$, $1-A^+(x)$>} and consequently $M_\alpha^\beta(A)$ produces an IFS, which has a non-zero hesitancy and hence is not a FS.

iii. When $\alpha = \beta$, (α,β)-multiplication reduces to α-multiplication.

iv. In general, (α,β)-multiplication produces an IFS even when A is an ordinary FS.

4.8.Properties. When A is an IFS, $M_\alpha^\beta(A)$ satisfies the following properties:

i. $M_\alpha^\beta(A) = A$ if and only if $\alpha = 1 = \beta$.

ii. When $\alpha < 1$ and $\beta = 1$, $M_\alpha^\beta(A) \subset A$.

Now we consider Defuzzification of FSs and Fuzzification of IFSs.

5.Defuzzification

In a fuzzy controller, the final option cannot be fuzzy. To communicate the option to the machine, it must be crisp. i.e., if the final conclusion is a fuzzy set B, then a crisp set defuzz(B) is formed, where 'defuzz' is a function mapping FSs to crisp ones.

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α-cuts and strong α-cuts produce crisp sets out of FSs. Another method is given by defuzzification operator $Defuzz^{MV}$ which identifies members with the highest membership value.

5.1. Definition[6]. Let $D = \{d_1, d_2, ..., d_k\}$ with membership functions $D^+: D \to [0, 1]$. The *maximum value defuzzification operator*, denoted by $Defuzz^{MV}$, is defined by

$$Defuzz^{MV}(D) = \{d_j : D^+(d_j) \geq D^+(d_i)\ \forall i \in \{1, 2, ..., k\}\}.$$

The following methods defined on fuzzy sets on **R** give singleton crisp sets as defuzzified value.

5.2. Definition [4]. Let A be a FS given by $A = \{(x_i, A(x_i)) / i = 1, 2, ..., n\}$ where $x_i \in$ **R**. Then the *centroid defuzzifier* δ is given by

$$\delta = \frac{\sum_{i=1}^{n} x_i A(x_i)}{\sum_{i=1}^{n} A(x_i)}$$

The centroid defuzzifier is the weighted average of the given values, weighted by the respective membership values.

5.3. Example. Let $X = \{1,2,3,4,5\}$ and $A = 0.8/1 + 0.6/2 + 0.8/3 + 0.3/4 + 0.1/5$

Then, $\delta = \frac{0.8 \times 1 + 0.6 \times 2 + 0.8 \times 3 + 0.3 \times 4 + 0.1 \times 5}{0.8 + 0.6 + 0.8 + 0.3 + 0.1} = 2.35$

Here $\delta \notin X$. Hence the value may be rounded off to a near value of elements of X. So, we take $\delta = 2$.

5.4. Definition [4]. Let A be a fuzzy set on **R** and let $C_m = \{x / A(x) = h(A)\}$, where h(A) denotes the height of A. Then the *center of maxima defuzzifier*, denoted by γ, is given by $\gamma = (\min C_m + \max C_m)/2$.

The center of maxima defuzzifier is the average of the lowest and highest of the elements with the largest membership grade. It identifies the central value among the values with the highest membership grade.

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5.5. Example. Let A = $0.5/a_1 + 1/a_2 + 0.5/a_3$. Then, $\gamma = a_2$, since $C_m = \{a_2\}$.

5.6. Example. Let A = $0.5/a_1 + 1/a_2 + 1/a_3 + 0.2/a_4$. Then, $\gamma = (a_2 + a_3)/2$, since $C_m = \{a_2, a_3\}$.

5.7. Definition [4]. Let $C_m = \{x/ A(x) = h(A)\}$, where h(A) is the height of A, where C_m is a finite set. Then, the *mean of the maxima defuzzifier*, denoted by η, is defined by $\eta = (1/k) \sum_{i=1}^{k} x_i$, where $x_i \in C_m$ and $k = |C_m|$.

The mean of the maxima is the average of the values with the highest membership grade.

5.8. Example. Let A = 0.8/1 + 0.6/2 + 0.8/3 + 0.8/4 + 0.2/5. Then $\eta = (1+3+4)/3 = 2.6 \approx 3$.

5.9. Remark. The center of maxima and the mean of the maxima defuzzifiers may not give a value in the universe. In this case we may approximate it to an appropriate value as in the case of the centroid defuzzifier.

In the next section we discuss IFS-to-FS conversions.

6. Fuzzification of IFS

Fuzzification of IFSs is the process of converting an IFS into a FS by eliminating the hesitancy grade. We have proposed several methods of fuzzification of IFSs [1].

In this section, we denote IFSs by A, B, ... and the corresponding fuzzified sets by $\widetilde{A}, \widetilde{B}$, ... First, we consider singleton IFSs and then extend it to finite IFSs.

In what follows, a singleton IFS A = $\{<x, A^+(x), A^-(x)>\}$ will be denoted as A = $<x, A^+(x), A^-(x)>$.

6.1. Method 1 (Arbitrary Allocation of Hesitancy)

Here, the hesitancy part is assigned to any of the grades A^+ or A^-.

6.2. Method 2 (Assigning Hesitancy to the Major Grade)

In this method, the hesitancy membership A^0 is added to the larger of the two grades A^+ and A^-.

6.3. Example. Let A = <x, 0.5, 0.3> be a given IFS. Then, the corresponding FS, fuzzified by the above method is, $\tilde{A}$ = <x, 0.7>.

6.4. Example. Let A = <x, 0.4, 0.5>. Then, $\tilde{A}$ = <x, 0.4>.

6.5. Method 3 (Equal Distribution of Hesitancy)

Here, the hesitancy grade is divided equally among the membership and non-membership grades.

6.6. Example. Let A = <x, 0.5, 0.3>. Then $\tilde{A}$ = <x, 0.6, 0.4> or, $\tilde{A}$ = <x, 0.6>

6.8. Example. Let A = <x, 0.6, 0.3>. Then, $\tilde{A}$ = <x, 0.65, 0.35> or, $\tilde{A}$ = <x, 0.65>

6.9. Method 4 (Proportionate Allocation)

Here, A^0 is divided in the proportion of A^+ and A^-.

i.e., $\tilde{A}^+ = A^+ + A^0 (A^+/(A^+ + A^-))$ and $\tilde{A}^- = A^- + A^0 (A^-/(A^+ + A^-))$.

6.10. Example. Let A = <x, 0.4, 0.2>.

Then, $\tilde{A}^+ = 0.4 + 0.4 (0.4/(0.4+0.2)) = 0.666$, and $\tilde{A}^- = 0.2 + 0.4 (0.2/(0.4+0.2)) = 0.334$

i.e., $\tilde{A}$ = <x, 0.666, 0.334>

We generalize the above methods to obtain a (continuous) class of fuzzification functions as follows.

6.11. Method 5(Weighted Proportionate Allocation)

Here, the hesitancy is allocated to A^+ and A^- in the proportion λ: $(1-\lambda)$, where $\lambda \in [0, 1]$.

i.e., If A = <x, A^+, A^->, then, $\tilde{A}$ = <x, $A^+ + \lambda A^0$, $A^- + (1-\lambda) A^0$>

6.12. Example. Suppose, A^+ denotes the proportion that favours a particular concept and A^-, the proportion that opposes it. Then, A^0 denotes the undecided segment. Now, suppose that 70% of the undecided segment may later adopt the

concept, then we may take $\lambda = 0.7$. Consequently we have, $\tilde{A} = <x, A^+ +$
$0.7A^0, A^- + 0.3A^0)>$.

6.13. Remarks. Method 5 proposed above generalises the previous methods.

i. When λ switches between 0 and 1 we get method 1.

ii. When $\lambda = 1$, the hesitancy is added to membership totally and when $\lambda =$ 0, the hesitancy is added to non membership totally. These two cases correspond to method 2.

iii. When $\lambda = 1/2$, the hesitancy is distributed to both the grades equally This corresponds to method 3.

iv. Taking $\lambda = A^+/(A^+ + A^-)$ we get method 4.

The methods discussed above relate to singleton IFSs only. They may be extended to arbitrary IFSs, by point-wise fuzzification. We illustrate the method below.

6.14. Example. Let A be a finite IFS consisting of three elements, given by

$$A = \{<a_1, 0.6, 0.3>, <a_2, 0.5, 0.3>, <a_3, 0.4, 0.4>\}.$$

Then, applying the second method, viz. equal distribution of hesitancy, we get the FS

$$\tilde{A} = \{<a_1, 0.65, 0.35>, < a_2, 0.6, 0.4>, < a_3, 0.5, 0.5>\}.$$

$$\text{i.e., } \tilde{A} = \{<a_1, 0.65>, < a_2, 0.6>, < a_3, 0.5, >\}.$$

7.Conclusion

Fuzzy sets were introduced as generalizations of crisp sets and intuitionistic fuzzy sets are generalizations of Fuzzy sets. However, some situations may require that one type be converted to another. Here, we have discussed methods to convert crisp sets to fuzzy sets and fuzzy sets to intuitionistic fuzzy sets and vice versa. By suitably combining these, any type of conversion is possible.

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