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FUZZY θ −SEPARATING COVER OF A FUZZY TOPOLOGICAL SPACE

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Abstract: The relationship between a $\boldsymbol{\theta}$ −separating cover and a G_δ −diagonal was given by R.E.Hodel in his paper 'Moore Space and $\boldsymbol{\omega\Delta}$ space' published in Pacific Journel of Mathematics in 1971.In this paper we review some basic concepts of topological spaces and fuzzify the above relationship.

Key Words: G_δ- set, G_δ- diagonal, $\boldsymbol{\theta}$ −separating cover, fuzzy topological space, fuzzy$\boldsymbol{\theta}$ −separating cover.

1. Introduction

Topology as a well-defined mathematical discipline, originated in the early part of the twentieth century. The word 'Topology' was derived from two Greek words, *topos* meaning 'surface' and *logos* meaning 'discourse' or 'study'.

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Topology thus literally means the study of surfaces. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. Topology is the mathematical study of the properties that are preserved through deformations, twistings, and stretching's of objects. In the mathematical field of topology, a **G_δ** set is a subset of a topological space that is a countable intersection of open sets. The notation originated in Germany with G for Gebiet (German: area or neighbourhood) meaning open set in this case and δ for Durchschnitt (German: intersection)

Fuzzy set theory and fuzzy logic were introduced by L. A. Zadeh in 1965 [8]. The characteristic function of a crisp set assigns a value of either 0 or 1 to each individual element in the universal set. 1 indicates membership and 0 non-memberships. The membership function of a fuzzy set assigns values in the interval [0,1] to the individual elements of the universal set (which is always a crisp set) and hence makes provision for partial membership also. Here we fuzzify the relationship between θ −separating cover and a G_δ−diagonal.

2. Preliminary Concepts

2.1. Definition. A ***topological space*** is an ordered pair (X, τ), where X is a set and τ is a family of subsets of X, satisfying the following axioms:

1. The empty set and X itself belong to τ.
2. τ is closed under arbitrary unions,
3. τ is closed under finite intersections

The elements of τ are called ***open sets*** and the collection τ is called a ***topology** on X*.

Let (X, τ) be a topological space. Then, a subset A of X is said to be ***closed*** in X if and only if its complement Ac is open in X.

2.2. Example.

1. A set X and the collection τ = P(X) (the power set of X), (X, τ) is a topological space. τ is called the ***discrete topology***.

2. Let X=R and let τ be the collection of subsets U of R such that for every $x \epsilon U$, there exist $r > 0$ such that the semi-open interval $[x, x + r)$ is contained in U. Then τ is a topology on R and is called the ***semi open interval topology.***

2.3. Definition. We call a subset $\mathcal{B}$ of τ as the ***Base for the topology*** if every set in τ can be obtained by union of some elements of $\mathcal{B}$.

2.4. Definition. A family $\mathcal{U}$of sets is said to be a ***cover*** C of A if A is contained in the union of members of $\mathcal{U}$. A ***subcover***$\mathcal{V}$ of $\mathcal{U}$ is a subfamily of $\mathcal{U}$ which itself is a cover of A.

If we are in a topological space then a cover is said to be open if each of its members is an open set.

2.5. Definition. Let X be a set and $\mathcal{G}$ a cover of X, x an element of X. The ***star of*** x with respect to $\mathcal{G}$, denoted$\boldsymbol{st}(\boldsymbol{x}, \boldsymbol{\mathcal{G}})$ is the union of all elements of $\mathcal{G}$ containing x. The order of x with respect to $\mathcal{G}$ denoted by $\boldsymbol{ord}\ (\boldsymbol{x}, \boldsymbol{\mathcal{G}})$is the number of elements of $\mathcal{G}$ containing x.

2.6. Definition. A **G_δ–set** is a subset of a topological space that is a countable intersection of open sets.

2.7. Definition. A space X has a **G_δ-*diagonal*** if its diagonal $\{(x,x): x \text{ in } X\}$ is a G_δ subset of $X \times X$

2.8. Definition. A *θ-separating cover* of a topological space X is a sequence $\mathcal{G}_1, \mathcal{G}_2 \ldots$ of open collections such that for any two distinct points x, y *in* X, there is an $n \in N$ such that

(a) $Ord\ (x, \mathcal{G}_n)$ is finite.

(b) There exists a G in $\mathcal{G}_n$, such that $x \in G$ and $y \notin G$

2.9. Theorem. Let X be a space with a θ-separating cover. If every closed subset of X is a G_δ, then X has a G_δ diagonal.

Proof: Let $\mathcal{G}_1$, $\mathcal{G}_2$… be fuzzy θ-separating cover of X. For each pair of positive integers n & k, let

$$\mathcal{H}_{nk} = \left\{H: H \neq \varphi, H = \cup_{i=1}^{k} G_i\ G_1, G_2 \ldots G_k \text{are distinct elements in } \mathcal{G}_n\right\}$$ and

$\mathcal{F}_{nk} = X - \cup\{H: H \in \mathcal{H}_{nk}\}$.

Now F_{nk} is closed and so $F_{nk} = \cup_{i=1}^{\infty} W_{nkj}$ where W_{nkj} are open sets.

For $j = 1,2,3$ ....Define $\mathcal{K}_{nkj} = H_{nk} \cup \{W_{nkj}\}$

Then each $\mathcal{K}_{nkj}$ is an open cover of X and sequence $\{\mathcal{K}_{nkj}; n, k, j \text{ in } N\}$ exhibits the G_δ diagonal property for X ■

3. Fuzzy set

3.1 Definition: Let X be any set. Then a **fuzzy set**A on X is defined by A = {(x, μ_A(x))/x∈ X}.Here μ_A is called the membership function and μ_A (x) is the degree of membership of x in the fuzzy set A.

3.2 Basic Operations on Fuzzy sets

The **standard complement**of a fuzzy set A, denoted by Ā, is the fuzzy set defined by

$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \forall x \in X$.

The **standard union** of two fuzzy sets A and B, denoted by A ∪B, is the fuzzy set defined by

$\mu_{(A \cup B)}(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X$.

The **standard intersection** of two fuzzy sets A and B, denoted by A ∩ B, is the fuzzy set defined by $\mu_{(A \cap B)}(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X$.

Let A and B be two fuzzy sets, then $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x), \forall x \in X$.

3.3. Definition: For $\alpha \epsilon [0,1], x \in X$, a **fuzzy point**$x_\alpha$ is defined to be the fuzzy set on X defined by

$$x_\alpha(y) = \begin{cases} \alpha & if \ y = x \\ 0 & if \ y \neq x \end{cases}$$

3.4. Remark 1: The set of all fuzzy sets on X is denotedby I^X. The fuzzy set which takes every element in X to 0 is denoted by **0** and which takes every element in X to 1 is denoted by **1** .

3.5. Remark 2:A fuzzy point $x_\alpha$is said to be in ȧ fuzzy set A denoted by $x_\alpha \in A$ if and only if $\alpha \leq A(x)$

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3.6. Definition: A family $\tau \subseteq I^X$ is called a **fuzzy topology** for X if it satisfies the following three axioms:

1. $\mathbf{0}, \mathbf{1} \in \tau$
2. $\forall A, B \in \tau \Rightarrow A \wedge B \in \tau$
3. $\forall (A_j)_{j \in J} \in \tau \Rightarrow \vee_{j \in J} A_j \in \tau$

The pair (X, τ) is called a fuzzy topological space. The elements of τ are called fuzzy open sets. A fuzzy set K is called fuzzy closed if $K^c \in \tau$

3.7. Example: Let $X = \{a, b, c\}$. If $\tau = \{\mathbf{0}, \mathbf{1}\ A, B, C\ , D\}$ is a collection of fuzzy sets on X as follows:

$A(a) = 0.3\ , A(b) = 0.1, A(c) = 1.$

B(a)=0.2 , B(b)=0.6 , B(c)=0

C(a)=0.3, C(b)=0.6 , C(c)=1

D(a)=0.2 , D(b)=0.1 , D(c)=0 , then (X, τ) is a fuzzy topological space.

3.8. Definition (Fuzzy Cover of fuzzy set): Let (X, τ) be a fuzzy topological space. A family $\mathcal{A}$ of fuzzy sets is a fuzzy cover of a fuzzy set A if and only if $\leq \vee\{A/A \in \mathcal{A}\}$. $\mathcal{A}$is an open cover if each A in $\mathcal{A}$ is open.

$\mathcal{A}$is a fuzzy cover of the space X if and only if $\vee\{A/A \in \mathcal{A}\} = \mathbf{1}$

3.9. Definition (Star of a fuzzy pointx_α): Let $\mathcal{G}$ be a cover of a fuzzy topological space (X, τ). For α∈(0,1] and a fuzzy point x_α. Then ***star of*** x_α with respect to $\mathcal{G}$, $\mathbf{st}(x, \mathcal{G})$ is defined by $st(x_\alpha, \mathcal{G}) = V\{B: B \in \mathcal{G}\ and\ B(x) \geq \alpha\}$

The order of x with respect to , denoted by $\boldsymbol{ord}(x, \mathcal{G})$, is the number of elements of $\mathcal{G}$ containing x.

3.10. Example: For the fuzzy topological space (X, τ) considered above, let $\mathcal{G} = \{A, C, 1\}$is an open cover of X. Then for the fuzzy point $a_{0.25}$, st($a_{0.25}$, $\mathcal{G}$)= V{B: B ∈$\mathcal{G}$ and B(α) ≥ 0.3}=A *V* C *V* 1=1

3.11. Definition: Let (X, τ) be a fuzzy topological space. Then a fuzzy set A on X is called a **G$_\delta$–set** if $\boldsymbol{A} = \wedge_{i=1}^{\infty} \boldsymbol{A_i}$ for $\boldsymbol{A_i} \in \tau$

3.12. Definition: A sequence ($\mathcal{A}n$) of open covers of a fuzzy topological space $(X, \tau)$is called a **G$_\delta$ -diagonal** sequence if for each x, y∈X , x≠y, α,β∈(0,1], there exist$n \in N$with $y_\beta \nleq st\ (x\ , \mathcal{A}n)$. ie, $\wedge_n st(x_\alpha, \mathcal{A}n) = x_\alpha$

A space $(X\ , \tau)$has a **G$_\delta$** -diagonal if there exists a **G$_\delta$** -diagonal sequence.

4. Fuzzy θ-separating cover.

4.1. Definition. ***Fuzzy θ-separating cover*** of a fuzzy topological space X is a sequence $\mathcal{G}_1$, $\mathcal{G}_2$... of fuzzy open collections such that for any two distinct fuzzy points x_α, y_β in X, there is an n ∈ N such that

(a) $Ord\ (x_\alpha, \mathcal{G}_n)$ is finite.

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(b) There exists a G in $\mathcal{G}_n$, such that $x_\alpha \in G$ and $y_\beta \notin G$

4.2. Theorem. Let X be fuzzy topological space with fuzzy θ-separating cover. If every fuzzy closed subset of X is a G_δ, then X has a G_δ diagonal.

Proof: Let $\mathcal{G}_1, \mathcal{G}_2 \ldots$ be fuzzy θ-separating cover of X. For each pair of positive integers n & k, let

$$\mathcal{H}_{nk} = \left\{ \begin{matrix} H: H \neq \varphi, H = \wedge_{i=1}^{n} G_i \, G_1, G_2 \ldots G_k \text{are distinct elements in } \mathcal{G}_n \end{matrix} \right\}$$ and

$$F_{nk} = 1 - \vee\{H: H \text{ in } \mathcal{H}_{nk}\}.$$

H is fuzzy open since $\mathcal{G}_1, \mathcal{G}_2 \ldots$ are fuzzy open collections. Therefore F_{nk} is fuzzy closed. Therefore $\mathcal{F}_{nk}$ is an intersection of countable collection of open sets since every fuzzy closed subset is G_δ.

(i.e.) $F_{nk} = \wedge_{j=1}^{\infty} W_{nkj}$ where W_{nkj} are open sets.

Let $\mathcal{K}_{nkj} = \boldsymbol{H}_{nk} \vee \{\boldsymbol{W}_{nkj}\}$. Then each $\boldsymbol{K}_{nkj}$ is a fuzzy open cover of X.

Since $\mathcal{G}_1, \mathcal{G}_2 \ldots$ is a fuzzy θ-separating cover, there exist an $n \in N$ such that $G \in \mathcal{G}_n$ and $x_\alpha \in G$ and $y_\beta \notin G$ for any distinct points x, y in X and $\alpha, \beta \in [0,1]$

Let $\mathcal{G}_1$ has this property. ie, there exist a $G \in \mathcal{G}_1$, such that $x_\alpha \in G$ and $y_\beta \notin G$.

$\Rightarrow y_\beta$ is not an element of any element in H_{1k} containing x_α

$\Rightarrow$ In $\mathcal{K}_{1kj}$ any element containing x_α does not contain y_β

$\Rightarrow y_\beta \notin st(x, \mathcal{K}_{1kj}) \Rightarrow \{\mathcal{K}_{nkj}: n, k, j \in N\}$ satisfies G_δ diagonal property.

Therefore X has a G_δ diagonal■

5. Conclusion

In this paper we discussed some of the concepts of topological spaces and fuzzy topological spaces. Then we defined the fuzzy θ-separating cover and proved a theorem based on this concept.

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