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FUZZY MAXIMAL G-MODULES

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Abstract: In this paper, we introduce fuzzy Maximal G-module on finite dimensional G-modules. Corresponding to any ascending chain of G-submodulesof a finite dimensional G-modules which terminates at some positive integer 'r' (Noetherian G-module), we have infinitely many ascending chain of fuzzy G-submodules, which also terminates at 'r' [6]. Here we prove equivalence conditions of such an ascending chain of fuzzy G-modules and a fuzzy maximal G-module.

Key words: Fuzzy set, fuzzy G-module, fuzzy Noetherian G-module, fuzzy maximal G-module

1. Introduction.

In representation theory, we consider the embedding of a finite group into a linear group. Here we consider those finite groups, which can be embedded in a ***finite*** linear group. The fuzzy set theory was introduced by L.A. Zadeh in 1965.After this Rosenfield started fuzzification of algebraic structures. As a continuation of these works the concept fuzzy maximal G-module is introduced and analysed here.

2. Preliminaries

2.1. Definition [1]. Let G be a finite group. M be a vector space over K (a subfield of C) and GL(M) be the group of all linear isomorphisms from M onto itself. A ***linear representation*** of G with representation space M is a homomorphism T : G → GL(M).

2.2. Example. Let G = S_n, the symmetric group with n symbols. Let M be an n-dimensional vector space over K with basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then M = $K_{\alpha_1} \oplus K_{\alpha_2} \oplus \ldots \oplus$
K_{α_n}.
Let $\sigma \in S_n$, then the map P(σ) : M → M defined by P(σ) (α_i) = $\alpha_{\sigma(i)}$ is an isomorphism from M in to M.
Then the function T: G →GL(M) defined by T(σ) = P(σ) is a homomorphism and hence a linear representation of G ■

2.3. Defenition[5]: Let M be a G-module. The G-submodules of M are said to satisfy the ascending chain condition (A.C.C.) if any chain of G-submodules of M ,$M_1 \subseteq M_2 \subseteq M_3 \subseteq$ ………….. terminates. That means there exist a positive integer 'r' such that $M_r = M_{r+1} = M_{r+2} = \ldots\ldots\ldots\ldots$
If G-submodules of M satisfy the A.C.C. then we shall call M a **Noetherian** G-module.

2.4. Example. The set **Q**(√2,√3) is the field obtained by adjoining the real numbers √2,√3 to **Q**.
Then M = **Q**(√2,√3) is a vector space over **Q** and the set { 1,√2,√3,√6} is a basis for **Q**(√2,√3) over **Q**. Let G {1, -1} Then M is a G=module.
Possible ascending chains of M are either the following chains or the one, which is obtained from these 2 chains.
M_1= {0} ⊆ M_2=**Q**(√2) ⊆ M_3 = **Q**(√6) ⊆ M_4 = **Q**(√2,√6) = M or
M_1= {0} ⊆ M_2=**Q**(√3) ⊆ M_3 = **Q**(√6) ⊆ M_4 = **Q**(√2,√6) = M
These two chains terminates at r = 4 and any other chain terminates at M_i = **Q**(√2,√6) for some i ε {1, 2, 3,4}■

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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2.8 Remark. From theorem 2.7, we have any vector space has a basis. So in the view of an infinite dimensional vector space, we have a basis $\{\alpha_1, \alpha_2, \ldots$ [illegible] $\}$. Also any chain $M_1 \subset M_2 \subset M_3 \subset$ ………………… in M also terminates as in the preceding proposition because M = span $\{\alpha_1, \alpha_2, \ldots$ [illegible] $\}$. But in this case we cannot find a positive integer 'r' where the chain terminates. So infinite dimensional G-modules are not Noetherian.

3. Fuzzy Noetherian G-module

3.1 Definition [2]. The characteristic function of a crisp set (classical set or ordinary sets) assigns a value of either 1 or 0 to each individual element in the universal set, thereby discriminating between members and non-members of the crisp sets under consideration. This function can be generalised in such a way that the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set in question. Larger values denote the higher degrees of the set membership. Such a function is called a ***membership function***, and the set defined by it a fuzzy set.

The most commonly used range of values of membership functions is the unit interval **[0,1]**, i.e. a fuzzy set μ on the set X is a function μ : X→[0,1].

3.2 Definition[4]. Let G be a finite group and M be a G-module over K, which is a subfield of complex numbers. Then a ***fuzzy G-module*** on M is a fuzzy subset μ of M such that

(i) $\mu(ax+by) \geq \mu(x) \wedge \mu(y), \forall$ a, b ε K and x, y ε M

and (ii) $\mu(gm) \geq \mu(m), \forall$ g ε G, m ε M. Where Λ is the minimum[infimum].

3.3 Example. Let G = {1,-1, i, -i}. Then M = C, the field of complex numbers is a G-module over itself. Define μ : M →[0,1] by

$\mu(x + iy) = 1,$ if $x = y = 0$

$= 1/2,$ if $x \neq 0, y = 0$

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$=\frac{1}{4}$, if $y \neq 0$

Then μ is a fuzzy G-module on M ■

3.4. Definition [4]. Let be μ, ν be fuzzy G-modules on M then $\mu \leq \nu$ if $\mu(x$
$\nu(x)$ for all $x \in M$.

3.5. Example. Let G = {1,-1, i, -i}. Then M = **C**, the field of complex numb
is a G-module over itself. Let the fuzzy G-module μ as in example 2.3
define $\nu : M \to [0,1]$ by

$$\nu(x + iy) = 1, \quad \text{if } x = y = 0$$

$$= 3/4, \quad \text{if } x \neq 0, y = 0$$

$$= 1/2, \quad \text{if } y \neq 0$$

Then μ and ν are fuzzy G-modules on M and $\mu(x) \leq \nu(x)$ for all $x \in M$. So
ν ■

3.6. Definition. Let G be a finite group and M be an n dimensional G-mod
over K, which is a subfield of complex numbers. Then afuzzy G-module μ on
is called a ***maximal*** if it satisfies the following

(i) the level cardinality of μ is greater than n

and (ii) $\nu \leq \mu$, for any fuzzy G-module ν on M

3.7. Example. Let G = {1,-1}. Then M = **C**, the field of complex numbers
G-module over the field **R**. Then M is 2 dimensional G-module. Define μ
$\to [0,1]$ by

$$\mu(x + iy) = 1, \quad \text{if } x = y = 0$$

$$= 0.999, \quad \text{if } x \neq 0, y = 0$$

$$= 0.998, \quad \text{if } y \neq 0$$ (here the membership grades is rounded to th
decimal places)

Then μ is a fuzzy G-module on M of cardinality 3 and for any fuzzy G-submodule ν on M, $\nu \leq \mu$. Then μ is a maximal fuzzy G-module ■

3.8. Proposition [4]. Any n-dimensional G-module has a fuzzy G-module with level cardinality n+1.

Proof: Let M be a G-module of dimension n over K. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of M over K.

Define $\nu : M \rightarrow [0,1]$ by

$\nu(c_1\alpha_1+c_2\alpha_2+..+c_r\alpha_n) = 1$, if $c_i=0$ for all i

$=\frac{1}{2}$, if $c_1\neq0$, $c_2=c_3=\ldots=c_n=0$

$=\frac{1}{3}$, if $c_2\neq0$, $c_3=c_4=\ldots=c_n=0$

$=\frac{1}{4}$, if $c_3\neq0$, $c_4=c_5=\ldots=c_n=0$

....

....

$=1/n-1$, if $c_{n-2}\neq0$, $c_{n-1}=c_n=0$

$=1/n$, if $c_{n-1}\neq0$, $c_n=0$

$=1/n+1$, if $c_n\neq0$

Then ν is a fuzzy G-module on M of level cardinality n+1 ■

3.9. Theorem[6]: Let M be an n-dimensional G-module and let the chain $M_1\subseteq M_2\subseteq M_3\subseteq$ .... of G-submodules of M terminates at 'r'. Then for k =1, 2, 3,...........there exist fuzzy G-modules μ_k on M, fuzzy G-submodules $\nu_k = \mu_k|_{M_k}$ on M_k such that the ascending chain $\mu_1\subseteq\mu_2\subseteq\mu_3\subseteq$ .... also terminates at r.

Proof: Let M be a G-module with a basis $\{\alpha_1, \alpha_2,\ldots\ldots,\alpha_n\}$. Then as proposition 1.11, for k = 1, 2, 3, …… M_k= span $\{\{\alpha_1, \alpha_2,\ldots\ldots,\alpha_k\}$. Then as the preceding proposition, the function ν_k: $M_k \rightarrow [0,1]$ defined by

ν_k $(c_1\alpha_1+c_2\alpha_2+..+c_r\alpha_k) = 1$, if c_i=0 for all i

$= 1/2$, if $c_1 \neq 0$, $c_2=c_3=\ldots\ldots=c_k=0$

$= 1/3$, if $c_2 \neq 0$, $c_3=c_4=\ldots\ldots=c_k=0$

$= 1/4$, if $c_3 \neq 0$, $c_4=c_5=\ldots\ldots=c_k=0$

……………………………………

$=1/k-1$, if $c_{k-2} \neq 0$, $c_{k-1}=c_k=0$

$=1/k$, if $c_{k-1} \neq 0$, $c_k=0$

$=1/k+1$, if $c_k \neq 0$

is a fuzzy G-module on M_k.

Let m = $^1/_{k+1}$. Then for each k, the function μ_k ; M $\rightarrow$ [0,1] defined by

μ_k (x) = ν_k(x), $\forall$ x ε M_k

= m, $\forall$ x ε M – M_k

is a fuzzy G-module on M and $\nu_k = \mu_k|_{M_k}$. Also the chain $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq$ ………. terminates at μ_r by the construction of μ_k■

3.10. Corollary [6]: Let M be an n-dimensional G-module and let the chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq$ ………… of G-submodules of M terminates at 'r'. Then there exist different (infinitely many) collection of chain of fuzzy G-submodules on M such that all these chains also terminates at r.

Proof: From theorem 3.9, for each k, there exists a fuzzy G-module ν_k on M_k. Let t ε (0,1], then replace '1'in the numerator of the definition of ν_k by 't' and the minimum 'm' by 's', where 0 < s < m < 1. Then corresponding to each pair (t, s), we get a chain $\{\mu_k\}_{(t,\ s)}$ of fuzzy G-submodules on M such that this chain terminates at 'r' ■

3.11. Theorem: Let M be an n-dimensional G-module. Then the following conditions are equivalent

(i) The chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq$ .... of G-submodules of M terminates at 'r'. Then for k =1, 2, 3,..........there exist fuzzy G-modules μ_k on M, fuzzy G-submodules $\nu_k = \mu_k|_{M_k}$ on M_k such that the ascending chain $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq$ .... also terminates at r.
(ii) The G-module M has a maximal fuzzy G-module.

Proof:

$(i) \Rightarrow (ii)$By theorem 3.10, corresponding to each pair (t, s), we get a chain $\{\mu_n\}_{(t,\ s)}$ of fuzzy G-submodules on M such that this chainterminates at 'r, where t and s defined in the theorem. Then for all ....$t_n < t_{n-1} < \dots\dots\dots < t_2 < t_1 = 1$, where $0 < t_i \leq 1$, the corresponding chain satisfy

$$\dots\dots\{\mu_n\}_{(t_n,\, s_n)} \subseteq \{\mu_n\}_{(t_{n-1},\, s_{n-1})} \subseteq \dots\dots \subseteq \{\mu_n\}_{(t_2,\, s_2)} \subseteq \{\mu_n\}_{((t_1,\, s_1)} = \{\mu_n\}_{((1,\ ^1/_{n+1})}.$$

Then $\mu = \{\mu_n\}_{((1,\ ^1/_{n+1})}$ is the maximal fuzzy G-module on M.

(ii) ⇒ (i) Assume that the G-module M has a maximal fuzzy G-module μ.

This implication is trivial because any ascending chain of fuzzy G-submodules$\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq$ ....terminates at μ.

4. Conclusion.

In this paper we introduce fuzzy Maximal G-module on finite dimensional G-modules. Also we prove equivalence conditions of an ascending chain of fuzzy G-modules and a fuzzy maximal G-module.

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