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FUZZY G-MODULES AND FUZZY REPRESENTATIONS

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Abstract : In this paper , we introduce and develop the notions of fuzzy G-modules and fuzzy representations as generalisation of G-modules and group representations in crisp module theory.

1. Introduction

The theory of group representations was developed by Frobenius.G in the last two decades of the 19th century. The works of E. Noether on representation theory led to the absorption of the theory of group representations into the study of modules over rings and algebras. The earlier treatment of group representation , heavily leaned on character theory. This , though simple and eloquent lacks depth. Many important results could be proved only for representations over algebraically closed fields. Module theoretic approach is better suited to deal with deeper results in representation theory. Moreover , module theoretic approach give more elegance to the theory. In particular, the G-module structure has been extensively used for the study of representations of finite groups.

Soon after the introduction of fuzzy set theory by L.A.Zadeh[2] in 1965, Rosenfield[1] initiated the fuzzification of algebric structures. As a continuation of these works, in this paper, we introduce fuzzy parallels of the notions of G-modules and group representations, and observe some of their basic properties.

2. Elementary Concepts

2.1. Notations.

M	: A vector space over a field K.
$Hom_K(M,M)$	: The set of all homomorphisms from M into itself.
K_n	: The set of all nxn matrices with entries from the field K.
GL(M)	: Invertible elements of $Hom_K(M,M)$.
GL(n,K)	: Invertible elements of K_n
$\vee$	: maximum[supremum]
$\wedge$	: minimum[infimum]

2.2. Definition. Let G be a finite group and M be a vector space over K. A ***linear representation*** of G with representation space M is a homomorphism T of G into GL(M).

2.3. Definition: A ***matrix representation*** of a finite group G of ***degree n*** is a homomorphism

$$T : G \to GL(n,K)$$

2.4. Example. Let G = (a) be a cyclic group of prime order p. Let K = Z_p. Then K is a field of characteristic p. Consider the matrix

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} ; \ 0, 1 \in K$$

It has the property that $X^p = I$, the identity matrix. Then the mapping $T : G \to GL(2,K)$ defined by

$$T(a^j) = X^j, \forall j = 0,1,2,\ldots\ldots,p-1$$

is a homomorphism ; and hence T is a matrix representation of order 2. □

2.5. Definition. Let G be a finite group and M be a vector space over K. Then M is called a ***G-module*** if there exists a map $(m,g) \to mg$ of M x G into M such that

$$m \,.\, 1_G = m, \forall\, m \in M, \quad 1_G \text{ being the identity in } G.$$
$$m(gh) = (mg)h, \forall\, m \in M;\ g, h \in G.$$
$$(k_1m_1 + k_2 m_2)g = k_1(m_1g) + k_2(m_2g), \forall\, k_1, k_2 \in K;\ m_1, m_2 \in M;\ g \in G$$

2.6. Remark. If M is a G-module, then the map $T : G \to GL(M)$ defined by

$$T(g) = T_g, \text{ where } T_g(m) = mg, (g \in G;\, m \in M)$$

is a homomorphism; and hence is a linear representation of G. T is said to be ***the representation afforded by the G-module M.*** Conversely, if G is a group and M is a vector space of dimension n over K, then any representation of G by n x n matrices induces a unique G-module structure on M. Hence ***linear representation*** , ***matrix representation*** and ***G-module*** are equivalent concepts.

2.7. Definition. If G is a group , then ***a fuzzy subgroup of G (or fuzzy group on G)*** is a function $\mu : G \to [0,1]$ such that for all $x, y \in G$

$$\mu(xy) \geq \mu(x) \wedge \mu(y)$$

and

$$\mu(x^{-1}) \geq \mu(x) \; .$$

2.8. Example. Consider the chain of groups $Z \supset 2Z \supset 4Z \supset 8Z \supset 16Z$.
Define $\mu : Z \to [\,0,1\,]$ by

$$\begin{aligned} \mu(x) &= 1 && \text{if} \quad x \in 16Z \\ &= 0.7 && \text{if} \quad x \in 8Z\text{-}16Z \\ &= 0.5 && \text{if} \quad x \in 4Z\text{-}8Z \end{aligned}$$

$$= 0.2 \quad \text{if} \quad x \in 2Z\text{-}4Z$$
$$= 0 \quad \text{if} \quad x \in Z\text{—}2Z$$

It can be easily verified that μ is a fuzzy subgroup of Z. □

2.9. Definition. A ***fuzzy subring*** of a ring R is a function $\mu : R \to [0,1]$ such that for all $x,y \in R$

$$\mu(xy) \geq \mu(x) \wedge \mu(y)$$
$$\mu(x\text{-}y) \geq \mu(x) \wedge \mu(y)$$

2.10. Example. Consider the ring $R = (Z_p, +_p, x_p)$ where p is a primenumber. Define $\mu : Z_p \to [0,1]$ by

$$\mu(x) = 1, \quad \text{if x is even}$$
$$= 0, \quad \text{if x is odd}$$

Then μ is a fuzzy subring of Z_p. □

2.11. Definition. If R is the field of real numbers and E is a vector space over R then a ***fuzzy vector subspace*** of E is a function $\mu : E \to [0,1]$ such that for all $a,b \in R$ and $x,y \in E$

$$\mu(ax + by) \geq \mu(x) \wedge \mu(y)$$

2.12. Example: R^n is a vector space over R. Define $\mu : R^n \in [0,1]$ by

$$\mu(x) = 0\ ; \ \text{if atleast one } x_j \neq 0, \text{ where } x = (x_j) \in R^n$$
$$= 1\ ; \ \text{if } x_j = 0, \forall j$$

Then μ is a fuzzy vector subspace of R^n. □

2.13. Definition: A ***fuzzy subalgebra*** of an algebra A over R is a function $\mu : A \to [0,1]$ such that

$$\mu(ax + by) \geq \mu(x) \wedge \mu(y)$$
$$\mu(xy) \geq \mu(x) \wedge \mu(y); \ \forall x, y \in A \text{ and } a, b \in R$$

Example 2.12 is a fuzzy subalgebra also.

3. Fuzzy G-modules and Fuzzy Representations

3.1. Definition. Let R be the field of real numbers and M be a vector space over R. Let G be a finite group and $T : G \to GL(M)$ be a representation of G in M. Then a ***fuzzy G-module*** is a function $\mu : M \to [0,1]$ such that

$$\mu(ax+by) \geq \mu(x) \wedge \mu(y) ; \forall x, y \in M ; a, b \in R$$

and

$$\mu(T_x(m)) \geq \mu(m) ; \forall m \in M ; x \in G$$

3.2. Example. Consider the G-module M = F(a) where G = (a) and F = R, the field of real numbers. Define $\mu : M \to [0,1]$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0 \\ 1/2, & \text{if } x \in F(a) - F \\ 0, & \text{if } x \in F - \{0\} \end{cases}$$

Then μ is a fuzzy G- module. □

3.3. Definition: Let μ be a fuzzy set in a set S and f is a function defined on S. Then the fuzzy set η in f(S) defined by

$$\eta(y) = \sup_{x \in f^{-1}(y)} \mu(x), \forall y \in f(S)$$

is called the ***image of μ under f*** and is denoted as $f(\mu)$.

3.4. Definition. If η is a fuzzy set in f(S), then the fuzzy set $\mu = \eta o f$ in S is called the ***pre-image of η under f***. It is denoted as $f^{-1}(\eta)$.

3.5. Proposition [1]. *Let $f : G \to H$ be a group homomorphism. Then for every fuzzy subgroup μ of G, $f(\mu)$ is a fuzzy subgroup of H.* □

3.6. Proposition [1]. *Let $f : G \to H$ be a homomorphism of G onto H. Then for every fuzzy subgroup η of H, $f^{-1}(\eta)$ is a fuzzy subgroup of G.* □

3.7. Homomorphism between Fuzzy Groups. Let G and H be groups, μ be a fuzzy group on G and ν be a fuzzy group on H . Let f be a group homomorphism of G onto H. Then f is called a ***weak fuzzy homomorphism of μ into ν*** if $f(\mu) \subseteq \nu$. We also say that μ is ***weak fuzzy homomorphic to*** ν and write $\mu \sim \nu$

The homomorphism f is called ***a fuzzy homomorphism of μ onto ν if $f(\mu)$*** $=$ ν we say that μ is homomorphic to ν and we write $\mu \approx \nu$

Let $f : G \rightarrow H$ be an isomorphism then f is a ***weak isomorphism*** if $f(\mu) \subseteq \nu$ and f is a ***fuzzy isomorphism*** if $f(\mu) = \nu$.

3.8. Example. Let G be the group (Z, +) and H be the multiplicative group {1, -1}.Define $f : G \rightarrow H$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is even.} \\ -1, & \text{if } x \text{ is odd} \end{cases}$$

Then f is a group homomorphism.Let μ be a fuzzy group on G defined by ;

$$\mu(x) = 1, \quad \text{if } x \text{ is even}$$
$$\mu(x) = 0, \quad \text{if } x \text{ is odd}$$

and ν be a fuzzy group on H defined by

$$\nu(1) = 1$$
$$\nu(-1) = 0$$

Then $\eta = f(\mu)$ is obtained as

$$\eta(1) = \sup_{x \in f^{-1}(1)} \mu(x) = 1$$

$$\eta(-1) = \sup_{x \in f^{-1}(-1)} \mu(x) = 0$$

Here $\eta = f(\mu) = \{ (1,1), (-1,0)\} = \nu$ and hence f is a fuzzy homomorphism of μ onto ν. □

3.9. Definition. Let G be a finite group, M be a vector space over K and $T : G \rightarrow GL(M)$ be a representation of G in M. Let μ be a fuzzy group on G and ν be a fuzzy group on the range of T . Then the representation T is a ***fuzzy representation*** if T is a fuzzy homomorphism of μ onto ν .

3.10. Example. Let $G = \{1, -1\}$ and M be a vector space over R.

Let $T : G \rightarrow GL(M)$ be defined by
$T(x) = T_x$, where $T_x(m) = mx$, $\forall x \in G$ and $m \in M$.

Then T is a representation of G in M.
Consider μ on G given by

$$\mu = \{(1,1), (-1,0)\}$$

and let v be a fuzzy subgroup on the range of T, viz. $\{T_1, T_{-1}\}$, defined by

$$\mu = \{(T_1, 1), (T_{-1}, 0)\}$$

Then $\eta(T_1) = \sup_{x \in T^{-1}(T_1)} \mu(x) = 1$, and

$$\eta(T_{-1}) = \sup_{x \in T^{-1}(T_{-1})} \mu(x) = 0$$

$\therefore \quad T(\mu) = v$

$\therefore$ T is a fuzzy representation. □

3.11. Theorem. *If $T : G \rightarrow GL(M)$ is a representation of G with representation space M, then for any fuzzy group μ of G, T is a fuzzy homomorphism of μ into $T(\mu) = v$.*

Proof. Follows from proposition 3.5. □

3.12. Theorem. *Let G be a finite group, M be a finite dimensional vector space over R and $T : G \rightarrow GL(M)$ be a representation of G. Let μ be a fuzzy group on G, v be a fuzzy group on the range of T and T be a fuzzy homomorphism of μ onto v. Then M admits a fuzzy G – module.*

Proof. Let $\dim(M) = n$ and $\{m_1, m_2, \ldots m_n\}$ be a basis for M. Then $\exists$ a unique $T_x \in GL(M)$ such that

$$T_x(m_i) = m_j \text{; where } 1 \leq i, j \leq n, x \in G.$$

Define η on M by

$$\eta(m) = \nu(Tx) \quad \text{if } m = 0$$
$$= \mu(x) \quad \text{if } m \neq 0$$

Then $\eta(am_1 + bm_2) \geq \eta(m_1) \wedge \eta(m_2)$; $\forall$ a,b $\in$ R if $m_1, m_2 \in M$

And $\eta(T_x(m)) \geq \eta(m)\ \forall\ x \in G$, $m \in \mu$

$\therefore$ η is a fuzzy G – module on M. Hence the theorem. □

References

[1] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35 (1971) 512-517.
[2] L.A.Zadeh, Fuzzy sets, Inform. Control, 8 (1965) 338 – 353.