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## RESEARCH IN MATHEMATICS

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# FUZZIFICATION OF FUNCTOR IN CATOGORY THEORY

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**Abstract:** The covariant functor (functor) plays an important role in Category theory. In this paper, by considering the extension principle in Fuzzy set theory, I have fuzzified this concept and proved that this fuzzified structure also satisfies the two axioms for the covariant functor.

**Key Words**: Category, Fuzzy Set, Extension Principle, Covariant functor.

## 1. Introduction

The history of category theory begins in 1945, when MacLane and Eilenberg jointly published the paper [3]. Category theory has come to occupy a central position in contemporary mathematics and theoretical computer science. It is considered as an alternative to set theory as a foundation for mathematics. It unifies mathematical structures. Almost all known mathematical structures with the appropriate structure-preserving map yields a category.

Fuzzy set theory and fuzzy logic were introduced by L. A. Zadeh in 1965 [8]. The characteristic function of a crisp set assigns a value of either 0 or

1 to each individual element in the universal set. 1 indicates membership and 0 non-membership. The membership function of a fuzzy set assigns values in the interval [0,1] to the individual elements of the universal set (which is always a crisp set) and hence makes provision for partial membership also. Fuzzy sets are useful for representing concepts with imprecise boundaries. Fuzzy sets are capable of expressing gradual transition from membership to non- membership and vice versa.

J.A. Goguen [5] considered a complete and distributive lattice L for the membership set, as replacement for [0,1], thus introducing the concept of an L-fuzzy set. In 1971 A. Rosenfeld [7] fuzzified the algebraic structures and proved a number of results.

## 2. Preliminary Concepts

**2.1. Definition.** A *category* $\mathcal{C}$ consists of:

(i) A class of objects, denoted as $ob\mathcal{C}$ and whose members are denoted as $A, B, C, \ldots$.

(ii) A family of mutually disjoint sets $\{mor(A, B)\}$ for all objects $A, B$ in $\mathcal{C}$, whose elements $f, g, h \in mor(A, B)$ are called morphisms and

(iii) A family of maps called composition $\{\ mor(A, B) \times mor(B, C) \rightarrow mor(A, C)\ \}$ in which $(f, g) \mapsto gf$ for all $A, B, C \in ob\mathcal{C}$

satisfying the following axioms:

(a) Associativity : For all $A, B, C, D \in ob\mathcal{C}$ and all $f \in mor(A, B)$, $g \in mor(B, C), h \in mor(C, D)$, we have $h(gf) = (hg)f$.

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(b) Identity: For each $A \in ob\mathcal{C}$ there is a morphism $I_A \in mor(A, A)$, called the identity, such that we have $fI_A = f$ and $I_A\, g = g$ for all $B, C \in ob\mathcal{C}$ and all $f \in mor(A, B)$ and $g \in mor(C, A)$.

**2.2. Example.** All sets together with the set maps and their composition form a category. This category is denoted by $\mathcal{Set}$.

**2.3. Example.** All groups together with group homomorphisms and their composition form a category. This category is denoted by $\mathcal{Grp}$.

**2.4. Definition.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $\mathcal{T}$ consist of

(i) a map $ob\mathcal{C} \ni A \mapsto \mathcal{T}(A) \in ob\mathcal{D}$.

(ii) a family of maps $\{mor_{\mathcal{C}}(A, B) \ni f \mapsto \mathcal{T}(f) \in mor_{\mathcal{D}}(\mathcal{T}(A), \mathcal{T}(B))\}$ for all $A, B \in ob\mathcal{C}$.

Then $\mathcal{T}$ is called a ***covariant functor*** (or simply a ***functor***) if $\mathcal{T}$ complies with the following axioms:

$\mathcal{T}(I_A) = I_{\mathcal{T}(A)}$for all $A \in ob\mathcal{C}$.

$\mathcal{T}(fg) = \mathcal{T}(f)\mathcal{T}(g)$for all $f \in mor_{\mathcal{C}}(B, C), g \in mor_{\mathcal{C}}(A, B)$ and for all $A, B, C \in ob\mathcal{C}$ .

**2.5. Example.** Consider the category $\mathcal{Grp}$. Its objects are sets with an additional structure. The morphisms are maps compatible with the structure of the sets. If one assigns to every object, the underlying set and to every morphism the underlying set map, then this defines a covariant functor from $\mathcal{Grp}$to$\mathcal{Set}$.

**2.6. Definition.** Given a set $X$, a ***fuzzy set*** on $X$ (or a ***fuzzy subset*** of $X$) is defined as a function $A : X \to [0,1]$. Its range is denoted as $Im(A)$.

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**2.7. Example.** Let $X = \{a, b, c, d, e\}$. Define $A : X \rightarrow [0,1]$ by $A(a) = 0.1$ ; $A(b) = 0.2$ , $A(c) = 0.3$, $A(d) = 0.4$, $A(e) = 0.5$. Then $A$ is a fuzzy subset of $X$.

**2.8. Definition.** Any function $f : X \longrightarrow Y$ induces two fuzzy functions $\mathbb{F} : \mathfrak{F}(X) \longrightarrow \mathfrak{F}(Y)$and $\mathbb{F}^{-1} : \mathfrak{F}(Y) \longrightarrow \mathfrak{F}(X)$(where$\mathfrak{F}(X)$and$\mathfrak{F}(Y)$denotethe collection of all fuzzy sets on $X$ and $Y$ respectively) defined as:

$$[\mathbb{F}(A)](y) = \sup\{A(x): f(x) = y\} \text{ and}$$

$$[\mathbb{F}^{-1}(B)](x) = B(f(x)) \quad .$$

This process of fuzzifying a function is called an ***extension principle***.

**2.9. Example.** Let X = {a,b,c} and Y = {1,2}, and a crisp functionf : a ↦ 1, b↦ 1, c↦2.

Let us consider A ∈ $\mathfrak{F}$ (X) and B ∈ $\mathfrak{F}$ (Y) defined as A = {(a, 0), (b , ·7), (c, ·9)} and B = {(1, ·3), (2, ·5)} respectively. When applying

$$[\mathbb{F}(A)](y) = \sup\{A(x): f(x) = y\} \text{for every } y \in Y \text{ and}$$

$$[\mathbb{F}^{-1}(B)](x) = B(f(x)) \text{for every } x \in X$$

to the function f, we obtain$\mathbb{F}$ (A) = {(1, ·7), (2, ·9)} and $\mathbb{F}^{-1}$ (B) = {(a, ·3), (b. ·3), (c. ·5)}. Such that $\mathbb{F}$ (A) ∈ $\mathfrak{F}$ (y) and $\mathbb{F}^{-1}$(B) ∈ $\mathfrak{F}$ (x)

## 3. Fuzzification of covariant functor

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{T}$ be a functor from $\mathcal{C}$ to $\mathcal{D}$. Then $\mathcal{T}$ consist of

(i) a map $ob\mathcal{C} \ni A \longmapsto \mathcal{T}(A) \in ob\mathcal{D}$.

(ii) a family of maps $\{mor_{\mathcal{C}}(A,B) \ni f \longmapsto \mathcal{T}(f) \in mor_{\mathcal{D}}(\mathcal{T}(A),\mathcal{T}(B))\}$ for all $A, B \in ob\mathcal{C}$.

Since $f \in mor_{\mathcal{C}}(A,B)$, $f : A \to B$ is a (crisp) function. Then by extension principle we can fuzzify this function as follows :

$\mathbb{F} : \mathfrak{F}(A) \longrightarrow \mathfrak{F}(B)$ such that $X \longmapsto \mathbb{F}(X)$ where $X : A \longrightarrow [0,1]$ (that is, $X$ is a fuzzy set on $A$). Then $\mathbb{F}(X)$ is a fuzzy set on $B$ ($\mathbb{F}(X): B \longrightarrow [0,1]$) by

$$[\mathbb{F}(X)](y) = \sup\{X(c): f(c) = y\},\ y \in B.$$

Then $\mathcal{T}(f): \mathcal{T}(A) \to \mathcal{T}(B)$ can be fuzzified by the map $\mathcal{T}(\mathbb{F}) : \mathfrak{F}(\mathcal{T}(A)) \longrightarrow \mathfrak{F}(\mathcal{T}(B))$ by $Z \longmapsto \mathcal{T}(\mathbb{F})(Z)$, where $Z$ is a fuzzy set on $\mathcal{T}(A)$, (that is, $Z : \mathcal{T}(A) \longrightarrow [0,1]$ ) and $\mathcal{T}(\mathbb{F})(Z)$ is a fuzzy set on $\mathcal{T}(B)$ defined by,

$$[\mathcal{T}(\mathbb{F})(Z)](k) = \sup\{Z(\mathcal{T}(x)): \mathcal{T}(\mathbb{F})(\mathcal{T}(x)) = k\}\ ,\ k \in \mathcal{T}(B).$$

Since $\mathcal{T}(I_A) = I_{\mathcal{T}(A)} \Rightarrow \mathcal{T}(\mathbb{I}_A) = \mathbb{I}_{\mathcal{T}(A)}$ and $\mathcal{T}(fg) = \mathcal{T}(f)\mathcal{T}(g)\ \mathcal{T}(\mathbb{F}\mathbb{G}) = \mathcal{T}(\mathbb{F})\mathcal{T}(\mathbb{G})$, the fuzzified covariant functor satisfies the conditions

(i) $\mathcal{T}(\mathbb{I}_A) = \mathbb{I}_{\mathcal{T}(A)}$ for all $A \in ob\mathcal{C}$.

(ii) $\mathcal{T}(\mathbb{F}\mathbb{G}) = \mathcal{T}(\mathbb{F})\mathcal{T}(\mathbb{G})$, for all $\mathbb{F} \in mor_{\mathcal{C}}(\mathfrak{F}(B), \mathfrak{F}(C))$, $\mathbb{G} \in mor_{\mathcal{C}}(\mathfrak{F}(A), \mathfrak{F}(B))$ and for all , $B, C \in ob\mathcal{C}$ .

## 4. Conclusion

In this paper we fuzzified the covariant functor by using Extension Principle and the fuzzified extension satisfies $\mathcal{T}(\mathbb{I}_A) = \mathbb{I}_{\mathcal{T}(A)}$ for all $A \in ob\mathcal{C}$ and

$\mathcal{T}(\mathbb{F}\mathbb{G}) = \mathcal{T}(\mathbb{F})\mathcal{T}(\mathbb{G})$, for all $\mathbb{F} \in mor_{\mathcal{C}}(\mathfrak{F}(B), \mathfrak{F}(C))$, $\mathbb{G} \in mor_{\mathcal{C}}(\mathfrak{F}(A), \mathfrak{F}(B))$ and for all $A, B, C \in ob\mathcal{C}$, corresponding to the condition $\mathcal{T}(I_A) =$

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$I_{\mathcal{T}(A)}$ for all $A \in ob\mathcal{C}$ and $\mathcal{T}(fg) = \mathcal{T}(f)\mathcal{T}(g)$ for all $f \in mor_{\mathcal{C}}(B, C), g \in mor_{\mathcal{C}}(A, B)$ and for all $B, C \in ob\mathcal{C}$. In the upcoming paper we consider the contra variant functor, try to fuzzify it by the inverse extension principle and study the properties of these two fuzzified structures.

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