Estimation of Time to Test Transform for Lomax Distribution Based on Records

Sowbhagya S Prabhu [1] and E. S. Jeevanand [2]

[1] Department of Mathematics, St. Albert's College, Ernakulam - 682018, India.

[2]Department of Mathematics, Union Christian College, Aluva, Ernakulam - 683102, India.

email: sprabhu@alberts.edu.in, radhajeevanand@gmail.com

Abstract: *In the present paper, we discuss the problem of estimation of TTT plot for the two parametric Lomax distribution. The estimation of TTT is obtained by employing the classical and Bayesian paradigm. Bayes estimators are obtained by using Gamma prior under SELF and LLF. Maximum likelihood estimation is also discussed. These methods are compared by using mean square error through simulation study with varying sample sizes.*

Keywords: Gamma prior, Lomax distribution, Loss function, Records, TTT

1. Introduction

The Lomax distribution also known as Pareto distribution of the second kind or Pearson Type VI distribution has been used in the analysis of income data and business failure data. The Lomax distribution was first introduced by K. S. Lomax in 1954 [18]. Corresponding p.d.f is given by

$$f(x:\theta,\lambda)=\frac{\theta\lambda^{\theta}}{(\lambda+x)^{\theta+1}}) \tag{1}$$

where $x, \theta, \lambda > 0$. Here θ, λ are shape and scale parameters respectively.

An application of the Lomax distribution in Receiver Operating Characteristic (ROC) was presented by Campbell and Ratnaparkhi [9]. Balakrishnan [6] and Ahsanullah [2] studied the distributional properties and recurrence relation moments of record values. Much work has been done with respect to the estimation of parameters using both classical and Bayesian techniques. Parametric and nonparametric inferences based on record values have also been studied extensively. Prabhu and Jeevanand [22] performs quasi Bayesian estimation of TTT in Lomax model.

The concept of Total Time on Test (TTT) transforms is well known for its applications in different fields of study such as reliability analysis, econometrics, stochastic modelling, tail orderings and ordering distributions. A major share of the literature on TTT is concerned with reliability problems that include characterization of ageing properties, model identification, tests of hypotheses, age replacement policies in maintenance, ordering life distributions, and defining new classes of life distributions.

Some of the applications of TTT were given in Bergman and Klefsjo [8], Bartoszewicz [7], Haupt and Schabe [15], Kochar et al. [16], Ahmed et al. [1], Li and Zou [17], Nanda and Shaked [20] and the references therein for further details. The estimation problem of TTT of the Lomax distribution discussed in Prabhu and Jeevanand [21].

Let F be a life distribution with finite mean µ. The scaled total time on test (TTT) transform of F is defined as,

$$\phi(t) = \frac{1}{\mu} \int_0^{F^{-1}(t)} \overline{F}(x)dx \qquad \text{for } 0 \leq t \leq 1$$

where $\overline{F} = 1 - F$ is the survival function with finite mean $\mu = \int_0^\infty \overline{F}(x)dx$.

Further more, let

$$F^{-1}(y) = inf\{x : F(x) \geq y\} \quad \text{ for } 0 \leq y \leq 1.$$

For the above model, the TTT simplifies to

$$\phi(t) = 1 - (1-t)^{\frac{\theta-1}{\theta}} \tag{2}$$

Record values and associated statistics are of interest and importance in several branches of science and social science such as psychology, medicine, pharmaceutical, engineering and pedagogy. Record values arise naturally in many applications involving data relating to weather, sports, economics and life testing studies. Many authors have studied records and their associated statistics as well as inference-based testing on records. Motivated by extreme weather conditions, Chandler [10] introduced record values and record value times. Feller [12] gave some examples of record values and record value times. Theory of record values and its distributional properties have been extensively studied in the literature. Sevgi, et al. [23] examined the relationship between order statistics and records. Galambose [13], Arnold and Balakrishnan [5], Ahsanullah [3], Arnold et al. [5] and Gulati and Padgett [14] have highlighted many of the advances in the theory of records and have mentioned several applications of record values.

Ahsanullah and Holland [4] discussed both scale and location estimation of the distribution of generalized extreme values based on records. Mathachan and Jeevanand [19] discuss the problem of estimation of entropy function of the geometric distribution based on record. The estimation problem of stress strength reliability for power function distribution is discussed in Dhanya and Jeevanand [11]. Let $r = (r_0, r_1, \cdots r_n)$ be a sequence of the first $(n+1)$ record values with $r_0 = x_1$ defined over the sample from Lomax distribution. Then the likelihood function of the sample of record values (R) from the population is given by,

$$l(R \setminus \theta, \lambda) = \prod_{i=0}^{n} \frac{f(R_i)}{\prod_0^{n-1}(1 - f(R_i))}$$

$$= \frac{\theta^{n+1}\lambda^{\theta}}{(r_n + \lambda)^{\theta+1}} \prod_{i=0}^{n-1} \frac{1}{r_i + \lambda}. \tag{3}$$

2. Estimation of TTT when λ is known

In this section we obtain the MLE and Bayes estimate of θ using record values when λ is known. The log likelihood function of the sample is

$$logl(R \setminus \theta, \lambda) = (n+1)log\theta + \theta log\lambda - (\theta+1)log(r_n+\lambda) - \sum_{i=0}^{n-1} log(r_i+\lambda). \tag{4}$$

Differentiating w.r.to θ and equating it with zero we get,

$$\frac{dlogl}{d\theta} = 0 \implies \frac{n+1}{\theta} = log(\frac{r_n+\lambda}{\lambda})$$

$$\hat{\theta} = \frac{n+1}{log(\frac{r_n+\lambda}{\lambda})} \tag{5}$$

The maximum likelihood estimator for the TTT denoted by $\hat{\phi}_{MLE}(t)$ can be obtained from (2) after replacing θ by $\hat{\theta}$.

The Bayesian approach provides the possibility for incorporating prior information about the relevant parameters. To this end, the parameter θ, is considered as a random variable, having some specified distribution. Here we suggest the conjugate prior distribution for the parameters and is given by

$$g(\theta) \propto \theta^{p-1} e^{-\theta\tau}, \text{with} \quad \theta, \tau, p > 0 \tag{6}$$

Combining (3) and (6) we get

$$f(\theta/R) \propto \frac{\theta^{n+1}\lambda^{\theta}}{(r_n+\lambda)^{\theta+1}} \prod_{i=0}^{n-1} \frac{1}{r_i+\lambda} \theta^{p-1} e^{-\theta\tau} \propto \theta^{R-1} e^{-\theta U} \tag{7}$$

where $R = n + p + 1$, $U = log(\frac{r_n+\lambda}{\lambda}) + \tau$. Let $\phi(t)$ be a parameter itself denoted by ϕ for simplicity. Replacing θ in (7) in terms of ϕ by that (2), we get the posterior of the TTT as

$$f(\phi \setminus x) = [C_1(t, 0)]^{-1} B_\phi^{R+1} e^{-B_\phi U} \frac{1}{1-\phi} \tag{8}$$

$$\text{where } C_1(t, d) = \int_0^t B_\phi^{R+d+1} e^{-B_\phi U} \frac{1}{1-\phi} d\phi, \qquad B_\phi = [1 - \frac{ln(1-\phi)}{ln(1-t)}]^{-1} \tag{9}$$

The symbol C with suffixes stands for the normalizing constants. Under squared error loss, the Bayes estimator of TTT is the mean of the posterior density given by

$$\hat{\phi}_{RS1} = E(\phi \setminus x) = \frac{C_1(t, 1)}{C_1(t, 0)}. \tag{10}$$

Under the LLF, the Bayes estimator for TTT is defined as

$$\hat{\phi}_{RL1} = \frac{-1}{a} ln\{E_{\phi \setminus x}(e^{-a\phi})\} = \frac{-1}{a} lnA_1 \tag{11}$$

Where $A_1 = \frac{\int_0^t B_\phi^{R+d+1} e^{-(a\phi + B_\phi U} \frac{1}{1-\phi} d\phi}{C_1(t,0)}$ and $C_1(t, d)$ is given in (9).

3. Bayesian estimation of TTT when λ is unknown

In this section we obtain the Bayes estimate of TTT when both the parameters are unknown. Here we take the joint prior distribution of θ and λ as

$$g(\theta, \lambda) \propto \frac{\theta^{p-1} e^{-\theta\tau}}{\lambda}, \qquad \theta, \tau, p > 0 \tag{12}$$

Combining the prior distribution (12) and the likelihood function (3), the posterior density of θ is derived as follows.

$$f(\theta \setminus x) \propto \theta^{\gamma-1} \int_0^\infty e^{-(\theta\varepsilon+B)} d\lambda \tag{13}$$

where $\gamma = n + p + 1, \varepsilon = \tau + log(\frac{r_n+\lambda}{\lambda})$ and

$$B = log(r_n + \lambda) + \sum_{i=0}^{n-1} log(r_i + \lambda) + log\lambda. \tag{14}$$

Replacing θ in (13) in terms of ϕ by that (2), we get the posterior of the TTT as

$$f(\phi \setminus \underline{x}) = \frac{R_\phi^{\gamma+1}(1-\phi)^{-1} \int_0^\infty e^{-(R_\phi\varepsilon+B)} d\lambda}{C_2(t,d)} \tag{15}$$

where

$$C_2(t,d) = \int_0^t \int_0^\infty \phi^d R_\phi^{\gamma+1}(1-\phi)^{-1} e^{-(R_\phi\varepsilon+B)} d\lambda d\phi]. \tag{16}$$

- The Bayes estimator of TTT under squared error loss function is given by

$$\hat{\phi}_{RS2} = E(\phi \setminus x) = \frac{C_2(t,1)}{C_2(t,0)} \tag{17}$$

Table 1: **Bias and MSEs (in paranthesis) of the estimate of ϕ using record values when λ is known for different values of θ**

n	θ	True ϕ	Bias $_{ML}$
40	1.5	0.112	0.2051
	2.5	0.193	0.1452
	3.5	0.225	0.1871
60	1.5	0.112	0.1358
	2.5	0.193	0.1075
	3.5	0.223	0.1185
80	1.5	0.112	0.1125
	2.5	0.193	0.0125
	3.5	0.223	0.0115

- The Bayes estimator of TTT under Linux loss function is given by

$$\hat{\phi}_{RL2} = \frac{-1}{a} ln\{E_{\phi\setminus x}(e^{-a\phi})\} = \frac{-1}{a} lnA_2 \tag{18}$$

where $A_2 = \frac{\int_0^t \int_0^\infty \phi^d R_\phi^{\gamma+1}(1-\phi)^{-1} e^{-(R_\phi \varepsilon + a\phi + B)} d\lambda d\phi}{C_2(t,0)}$ and $C_2(t,d)$ is given in (16)

Table 2: **Bias and MSEs (in paranthesis) of the estimate of ϕ using record values when λ is known for different values of θ**

n	θ	Bias $_{BS1}$	Bias $_{BL1}$
40	1.5	0.11056 (0.04582)	0.12018 (0.02563)
	2.5	0.193 (0.12562)	0.14738 (0.10052)
	3.5	0.225 (0.04875)	0.13178 (0.00357)
60	1.5	0.18713 (0.00251)	0.14447 (0.00605)
	2.5	0.18545 (0.15610)	0.14323 (0.00405)
	3.5	0.13648 (0.18001)	0.11990 (0.05623)
80	1.5	0.14418 (0.11125)	0.12780 (0.08520)
	2.5	0.14114 (0.25140)	0.12606 (0.00101)
	3.5	0.13346 (0.01542)	0.11858 (0.00152)

Table 3: **Bias and MSEs (in paranthesis) of the estimate of ϕ using record values when λ is unknown for different values of θ**

n	θ	Bias $_{BS2}$	Bias $_{BL2}$
	1.5	0.16037 (0.01025)	0.14056 (0.00223)
40	2.5	0.15185 (0.01425)	0.12766 (0.01024)
	3.5	0.12769 (0.14520)	0.11206 (0.10256)
	1.5	0.14608 (0.01470)	0.12475 (0.00356)
60	2.5	0.14462 (0.14258)	0.12351 (0.00025)
	3.5	0.11779 (0.01874)	0.10905 (0.01741)
	1.5	0.12659 (0.11245)	0.11804(0.05478)
80	2.5	0.13306 (0.14025)	0.12606 (0.07245)
	3.5	0.12602 (0.17802)	0.11858 (0.00475)

4. Simulation study

In this section, we study the performance of the estimators obtained so far using simulated data. The simulation encompasses the parameter values $\theta = 1.5, 2.5, 3.5$. We simulate 1000 samples of different sizes that have been generated and the above measures are calculated. The bias and MSEs of the different estimators are given in Tables 1, 2 and 3.

5. Conclusion

Based on the set of the upper record values the present paper proposed classical and Bayesian approaches to estimate the TTT for Lomax model. Comparisons are made between different estimators based on simulation study and the following were observed: (i) The Bayesian estimates are found to be better than the maximum likelihood estimates.

(ii) From the table we can observe that the estimate under squared error loss function has lesser bias than the Linex Loss function.

References

1. Ahmed I A, Li X, Kayid M, The NBUT class of life distributions, IEEE Trans. Reliab, Vol. 54, pp. 396-401, 2005.

2. Ahsanullah M, Record values of the Lomax distribution, Statistica Neerlandica, Vol. 45, pp. 21-29, 1991.

3. Ahsanullah M, Record statistics, Nova Science Publishers, New York, 1995.

4. Ahsanullah M, Holland B, On the use of record values to estimate the location and scale parameters of the generalized extreme value distribution, The Indian Journal of Statistics, Vol. 56, pp. 480-499,1994.

5. Arnold B C, Balakrishnan N, Nagaraja H N, Records, John Wiley and Sons, New York, 1998.

6. Balakrishnan M, Relations for single and product moments of record values from Lomax distribution, The Indian Journal of statisticas, Vol. 56, pp. 140-146, 1994.

7. Bartoszewicz J, Stochastic order relations and the total time on test transform, Statist. Prob. Lett, Vol. 22, pp. 103-110, 1995.

8. Bergman B, Klefsjo B, The total time on test concept and its use in reliability theory, Operat. Res, Vol. 32, pp. 596-606, 1984.

9. Campbell G, Ratnaparkhi M. V, An application of Lomax distribution in receiver operating characteristic (ROC) curve analysis, Communications in statistics, Vol. 22, pp. 1681-1687, 1993.

10. Chandler K N, The distribution and frequency of record values, Journal of the Royal statistical Society Series B, Vol. 14, pp. 220-228, 1952.

11. Dhanya M, Jeevanand E S, Stress strength reliability of power function distribution based on records, Journal of Statistical Application and Probability, Vo. 7(1), pp. 39-48, 2018.

12. Feller W, An introduction to probability theory and its applications, Wiley, New York, 1966.

13. Galambose J, The asymptotic theory of extreme order statistics, Second Edition, R.E. Krieger Pub. Co., 1987.

14. Gulati S, Padgett W J, Parametric and nonparametric inference from record breaking data, Springer-Verlag, New York, 2003.

15. Haupt E, Schabe H, The TTT transformation and a new bathtab distribution model, J. Statist. Planning Infer, Vol. 60, pp. 229-240, 1997.

16. Kochar S C, Li X, Shaked M, The total time test transform and the excess wealth stochastic orders of distribution, Adv. Appl. Prob, Vol. 34, pp. 826-845, 2002.

17. Li X, Zou M, Preservation of stochastic orders for random minima and maxima with applicationas, Naval Res. Logistics, Vol. 51, pp. 332-334, 2004.

18. Lomax K S, Business failures: Another example of the analysis of failure data, Journal of the American Statistical Association Vol 49, pp. 847–852, 1954.

19. Mathachan Pathiyil, Jeevanand E S, Estimation of the residual entropy function of the geometric distribution using record values, in the Proceedings of the National Workshop on Bayesian Statistics and MCMC Methods using BUGS and R, St. Thomas College, Pala, pp. 26-30, July, 2009.

20. Nanda A K, Shaked M, Partial ordering and aging properties of order statistics when sample size is random:a brief review. Commun. Statist. Theory Meth, Vol. 37, pp. 1710-1720, 2008.

21. Prabhu S S Jeevanand E S, Estimation of time to test transform for Lomax distribution, Far East Journal of Theoretical Statistics, Vol. 61(1), pp. 21-33, 2021.

22. Prabhu S S, Jeevanand E S, Quasi-Bayesian stimation of time to test transform in Lomax model, Proceedings of National Seminar in Applied statistics and Symposium on Stochastic, Modelling, NSASSSM-2020, pp. 233-240, 2020.

23. Sevgi Y O, Mohammad A, Fazil A A, Funda A, Switching record and order statistics via random contractions. Statistics p Probability Letters, Vol. 73, pp. 207-217, 2005.