VOLUME IX; NUMBER 2

DECEMBER 2010

THE ALBERTIAN JOURNAL OF PURE AND APPLIED MATHEMATICS

TAJOPAAM

A JOURNAL DEVOTED TO THE ENCOURAGEMENT OF

RESEARCH IN MATHEMATICS

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TAJOPAAM
Volume IX, Number 2
December 2010
Pages 201 - 213

CHARGED FLUID SPHERE OF CLASS ONE

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Abstract: Spherically symmetric space-time of dimension 4 is at most of class 2 i.e. it can be embedded in a flat space-time of dimension 6. However, if it is possible to embed a 4-dimensional Riemannian space-time into a 5 dimensional flat space-time, then it is said to be class one. Here, we obtain a spheroidal space-time's of class one which is suitable to describe a compact charged fluid sphere in equilibrium.

Key Words: Spherically symmetric space-time, flat space-time, Einstein's field equation, Maxwell's equation.

1. Introduction

Spherical stars in equilibrium state are electrically neutral objects. But, the electric force of repulsion may help to avert the collapse of a spherically symmetric distribution of matter to a point singularity. The gravitational attraction is then balanced by the combined effect of pressure gradient and the electrostatic repulsive force. These factors provide sufficient motivation for finding the interior sources for the Reissner-Nordstrom metric, which uniquely describes the space-time of a static spherically symmetric charge distribution. Reissner- Nordstrom metric found by Reissner (1916) and Nordstrom (1918) is a straight forward generalization of the Schwarzschild exterior metric. Subsequently many exact solutions of the coupled Einstein-Maxwell equations corresponding to spherical distributions of charged fluids have been reported. Papapetrou - Majumdar (1947) studied the charged dust in equilibrium in which the collapse to a singularity is averted by the presence of electrical charge. Bonnor (1960, 1965) showed that for spherical distributions of uniformly charged dust in equilibrium, the absolute value of charge density must be equal to matter density. De and Raychaudhari (1968) showed that the equality of the magnitudes of charge density and matter density ia s direct consequence of Einstein-

Maxwell equations. Cooperstock and de la Cruz (1978) studied relativistic spherical charged distributions of perfect fluid in equilibrium and obtained an explicit solution of Einstein-Maxwell equations. Their solution is a generalization of Schwarzschild interior solution, with matter density decreasing outward. Bonnor and Wickramasuriya (1975) have obtained a static interior dust metric with matter density increasing outward. Interior space-time metric for charged fluid sphere with uniform density was obtained by Kyle and Martin (1967) and Mehra and Bohra (1979). In all the above cases the physical three-space t=constant is spherical. Tikekar (1984), Patel and Pandya (1986), Patel and Koppar (1987) obtained interior Reissner-Nordstrom metrics in which the physical three space t = constant is spheroidal. Patel , Tikekar and Sabu (1997) established a theorem useful for generating new classes of interior Reissner-Nordstrom solutions from known solutions.

It is known that spherically symmetric space-time of dimension 4 is atmost of class 2 (Eiesland, 1925), i.e. it can be embedded in a flat space-time of dimension 6. However, if it is possible to embed a 4-dimensional Riemannian space-time into a 5 dimensional flat space-time, it is said to be class one. The Schwarzschild interior metric and the Robertson-Walker metric whose3-dimensional physical space have the geometry of a sphere are known to be embedding class one. Here, we obtain a spheroidal space-time's of class one which is suitable to describe a compact charged fluid sphere in equilibrium.

2. The Field Equations

We assume that the space-time of the spherical charged fluid distributions in equilibrium is the spheroidal space-time with the metric

$$ds^2 = \exp(\nu)\, dt^2 - \exp(\lambda)\, dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1}$$

with $e^\lambda = \frac{1-K\frac{r^2}{R^2}}{1-\frac{r^2}{R^2}}$, whose metric variable are related to physical variables through Einstein's field equation

$$R_i^j - \frac{1}{2}R\delta_i^j = -8\pi T_i^j \tag{2}$$

For charged fluids the energy – momentum tensor

$$T_i^j = (\rho + p)u_i u^j - p\delta_i^j + \frac{1}{4\pi}\left[-F_{i\alpha}F^{j\alpha} + \frac{1}{4}\delta_i^j F_{\alpha\beta}F^{\alpha\beta}\right] \tag{3}$$

F_{ij} are the components of electromagnetic field tensor which satisfy Maxwell's equations

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

and

$$\frac{\partial}{\partial x^k}\left(F^{ik}\sqrt{-g}\right) = 4\pi J^i\sqrt{-g} \tag{4}$$

$J^i = \sigma u^i$ denotes the four – current vector where σ is the charge density of the distribution. For charged fluid at rest

$$u^i=(0,0,0,e^{-\nu/2}). \tag{5}$$

The assumption of spherical symmetry of the space-time implies that the only surviving component of the electro-magnetic field tensor is F_{14} = -F_{41}. Maxwell's equation (4) for the space time metric (1) determines

$$F_{41} = -\frac{e^{\frac{\nu+\lambda}{2}}}{r^2}\int_0^r 4\pi\sigma r^2\, e^{\frac{\lambda}{2}}dr. \tag{6}$$

We introduce the electric field intensity E as

$$E^2(r) = -F_{41}F^{41}.$$

Then it follows from (6) that

$$4\pi\sigma = \frac{1}{r^2}\left[\frac{d}{dr}(r^2E)\right]e^{-\lambda/2}.$$

Subsequently, the quantity

$$q(r) = 4\pi\int_0^r e^{\lambda/2}\,\sigma r^2 dr \tag{7}$$

Represents the total charge contained within the sphere of radius r and the electric field intensity will have the expression

$$E(r) = \frac{q(r)}{r^2}. \tag{8}$$

In view of (3), the Einstein field equations (2) for the metric (1) reduce to the system of three equations:

$$8\pi\rho + E^2 = -e^{-\lambda}\left[\frac{1}{r^2} - \frac{\lambda'}{r}\right] + \frac{1}{r^2}, \tag{9}$$

$$-8\pi p + E^2 = -e^{-\lambda}\left[\frac{1}{r^2} + \frac{\nu'}{r}\right] + \frac{1}{r^2}, \tag{10}$$

$$-8\pi p + E^2 = -e^{-\lambda}\left[\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right]. \tag{11}$$

These three equations relate the four variables ρ, p, E^2, ν since the assumption of spheroidal geometry for the space-time fixes up e^{λ} as stated in (1). Specific solution of this system of equations can only be obtained when one or more relation between these variables is available.

3. Charged Fluid Sphere of Class One

Tikekar (Ph.D Thesis, 1975) has shown that, if the static sapce-time of the metric (1) is of embedding class one, then

$$e^{\lambda} = 1 + \alpha e^{\nu}\nu'^2 \tag{12}$$

where α is an arbitrary constant. The space-time of metric (1) will be a spheroidal space-time of embedding class one if and only if

$$e^{\lambda} = \frac{1 - K\frac{r^2}{R^2}}{1 - \frac{r^2}{R^2}} = 1 + \alpha e^{\nu}\nu'^2. \tag{13}$$

The above differential equation admits the general solution in the form

$$e^{\nu} = \left[A + B\sqrt{1 - \frac{r^2}{R^2}}\right]^2 \tag{14}$$

where A and B are arbitrary constants. In fact the constant B is related to α by the relation

$$B = \frac{-R\sqrt{1-K}}{\sqrt{\alpha}}.$$

Hence, spheroidal static space-time of embedding class one is described by the metric

$$ds^2 = \left[A + B\sqrt{1 - \frac{r^2}{R^2}}\right]^2 dt^2 - \frac{1-K\frac{r^2}{R^2}}{1-\frac{r^2}{R^2}} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{15}$$

When $K=0$, this metric can be identified as Schwarzschild interior metric describing the space-time in the interior of a fluid sphere with uniform matter density $\rho = \frac{3}{8\pi R^2}$ in equilibrium. If K is different from zero, the metric may represent the space-time of charged fluid distribution in equilibrium with matter density and fluid pressure given by

$$16\pi\rho = \frac{A\left[6-6K-K\frac{r^2}{R^2}+K^2\frac{r^2}{R^2}\right]+B\sqrt{1-\frac{r^2}{R^2}}\left[6-6K+K^2\frac{r^2}{R^2}\right]}{R^2\left(1-K\frac{r^2}{R^2}\right)^2\left[A+B\sqrt{1-\frac{r^2}{R^2}}\right]} \tag{16}$$

$$16\pi p = \frac{A\left[-2+2K+K\frac{r^2}{R^2}-K^2\frac{r^2}{R^2}\right]+B\sqrt{1-\frac{r^2}{R^2}}\left[-6+2K+4K\frac{r^2}{R^2}-K^2\frac{r^2}{R^2}\right]}{R^2\left(1-K\frac{r^2}{R^2}\right)^2\left[A+B\sqrt{1-\frac{r^2}{R^2}}\right]} \quad (17)$$

The electric field required to maintain the equilibrium of this distribution will be

$$2E^2 = \frac{K\frac{r^2}{R^2}\left[A(K-1)+B(K-2)\sqrt{1-\frac{r^2}{R^2}}\left[-6+2K+4K\frac{r^2}{R^2}-K^2\frac{r^2}{R^2}\right]\right]}{R^2\left(1-K\frac{r^2}{R^2}\right)^2\left[A+B\sqrt{1-\frac{r^2}{R^2}}\right]} \quad (18)$$

(16), (17) and (18) follows from the Einstein's field equations. When K = 0, the electric field switched off and the solution degenerates into Schwarzschild interior solution.

4. Boundary conditions and Physical Plausibility

In order to ensure the physical plausibility of the model, it is necessary to have information about the constants A and B which are determined by the conditions which the metric (15) is expected to fulfill at the boundary $r = a$ of the charged fluid distribution. At $r = a$ the metric to be continuous with the exterior Reissner-Nordstrom metric

$$ds^2 = \left[1-\frac{2m}{a}+\frac{q^2}{a^2}\right]\ dt^2 - \left[1-\frac{2m}{a}+\frac{q^2}{a^2}\right]^{-1} dr^2 - r^2(d\theta^2 sin^2\theta d\phi^2). \quad 19)$$

And that *p(a) = 0*. These conditions imply the relations

$$\left(1-\frac{2m}{a}+\frac{q^2}{a^2}\right)=\frac{1-\frac{a^2}{R^2}}{1-K\frac{a^2}{R^2}}=\left[A+B\sqrt{1-\frac{a^2}{R^2}}\right]^2 \tag{20}$$

and

$$A\left[-2+2K+K\frac{a^2}{R^2}-K^2\frac{a^2}{R^2}\right]+\sqrt{1-\frac{a^2}{R^2}}\left[-6+2K+4K\frac{a^2}{R^2}-K^2\frac{a^2}{R^2}\right]=0 \tag{21}$$

which determines the constants *A* and *B* and *m* as

$$A=\frac{\sqrt{1-\frac{a^2}{R^2}}\left[-6+2K+4K\frac{a^2}{R^2}-K^2\frac{a^2}{R^2}\right]}{\left(1-K\frac{a^2}{R^2}\right)^{1/2}\left[4-3K\frac{a^2}{R^2}\right]} \tag{22}$$

$$B=\frac{\left[-2+2K+K\frac{a^2}{R^2}-K^2\frac{a^2}{R^2}\right]}{\left(1-K\frac{a^2}{R^2}\right)^{1/2}\left[4-3K\frac{a^2}{R^2}\right]} \tag{23}$$

and
$$\frac{2m}{a}=\frac{(1-K)\frac{a^2}{R^2}}{1-K\frac{a^2}{R^2}}+\frac{q^2}{a^2} \tag{24}$$

where
$$\frac{q^2}{a^2}=\frac{K\frac{a^4}{R^4}\left[A(K-1)+K(K-2)\left(1-\frac{a^2}{R^2}\right)^{1/2}\right]}{2\left(1-K\frac{a^2}{R^2}\right)^2\left[A+B\sqrt{1-\frac{a^2}{R^2}}\right]} \tag{25}$$

with *q* denoting the total charge of the configuration.

After a straight forward but lengthy computation it is ensured that E^2 is positive throughout the configuration if $K < 0$.

The charged fluid distribution described by the metric (15) will be physically viable, if it fulfills the requirements $\rho > 0, p > 0, \rho - 3p > 0, E^2 > 0$ and that ρ, p and E^2 are regular throughout. Owing to the complexity of the expression for ρ and $p > 0$, it is not possible to examine whether all the above conditions are fulfilled throughout the configuration. We have examined the implications of those requirements at the centre. At the center r =0, where

$$8\pi\rho(0) = \frac{3(1-K)}{R^2} \tag{26}$$

$$8\pi p(0) = \frac{A[-1+K]+B[-3+K]}{R^2[A+B]}, \tag{27}$$

The positivity of density at the centre is ensured since $K < 1$. However the requirement p(0) > 0 is fulfilled if and only if

$$K = \frac{A+3B}{A+B}. \tag{28}$$

The fulfilment of $\rho(0) \geq 3p(0)$ at the centre lead to the condition $\frac{A+2B}{A+B} \geq K$. These conditions together imply

$$\frac{A+2B}{A+B} \geq K \geq \frac{A+3B}{A+B}. \tag{29}$$

The electric field intensity is zero at the centre. However, it is necessary that $E^2 > 0$ throughout. It is possible to estimate using numerical procedure the total mass, size and the total charge of the charged fluid sphere using the scheme similar to the one adopted by Tikekar (1984) for different values of $\frac{a^2}{R^2}$, once a specific choice is made for the curvature parameter K. In the following Table, we have presented the values of $\mu = \frac{\rho(a)}{\rho(0)}$, R, a, m, $\frac{q^2}{a^2}$, $\rho(0), p(0), and\ \rho(0) - 3p(0) > 0$ for models with different choices of $\frac{a^2}{R^2}$.

Table 1

$\frac{a^2}{R^2}$	μ	R	a	m	q	$p(0)$	$\rho(0)$	$\rho(0) - 3p(0)$
.08	.87	37.53	10.61	.8	.561	.00019	.00851	.00794
.16	.77	35.28	14.11	2.01	1.400	0.00049	0.00963	0.00816
.24	.69	33.31	16.32	3.31	2.289	0.00095	0.01081	0.00794
.32	.62	31.57	17.86	4.61	3.159	0.00166	0.01203	0.00702
.40	.56	30.03	18.99	5.84	3.982	0.00279	0.01330	0.00493
.48	0.51	28.65	19.85	7.00	4.750	0.00464	0.01461	0.00068
.52	.49	28.02	20.20	7.55	5.111	0.00604	0.01528	-00.00286

The value m and a are in agreement with the corresponding values expected of a neutron star. It is observed that for models with $\frac{a^2}{R^2} \leq 0.48$ the conditions $\rho(0), p(0), and\ \rho(0) - 3p(0) > 0$ are fulfilled at the centre.

5. Conclusion. Spheroidal space-time's of class one are suitable to describe the interior space-time of a compact charged fluid sphere in equilibrium.

6. References

[1]. Bonnor, W. B., *Z. Phys.*, 160, 59 (1960).

[2]. Bonnor, W. B.,*Mon. Not. Roy. Astron. Soc.*, 129, 443 (1965).

[3]. Bonnor, W. B., and Wickramasuriya, S. B. P., *Mon. Not. Roy. Astron. Soc.*, 170, 643 (1975)

[4]. Cooperstock, F. I., and de la Cruz , V., *Gen. Rel. Grav.*, 9, 835 (1978)

[5]. De, U. K., and Raychaudhari, A. K., *Proc. Roy. Soc. (London)*, A303, 97 (1968)

[6]. Eiesland, J., *Trans. Amer. Math. Soc.*, 27, 213 (1925)

[7]. Kyle, C. F., and Martin, A. W., *Nuovo Cim.*, 50, 583 (1967)

[8]. Majumdar, S. D., *Phys. Rev.*, 72, 390 (1947)

[9]. Mehra, A. L., and Bohra, M., *Gen. Rel. Grav.*, 11, 333 (1979)

[10]. Nordstrom , G., *Proc. Kon. Ned., Acad. Wet.*, 20, 1238 (1918)

[11]. Papapetrou, A., *Proc. R. Irish Acad.*, 191 (1947)

[12]. Patel , L. K., and Pandya, B. M., *Acta Physics Hungarica*, 60, 57 (1986)

[13]. Patel, L. K., and Koppar, S. S., *Aust. J. Physics.*, 40, 441 (1987)

[14]. Patel ,L. K., Tikekar, R., and Sabu, M. C., (1997), *Gen. Rel. Grav.*,

[15]. Reissner , H., *Annalen Physik*, 50, 106 (1916)

[16]. Tikekar , R., Ph D Thesis (1975)

[17]. Tikekar , R., *Gen. Rel. Grav.*, 16, 445 (1984)