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CHAIN OF ILSG'S IN CYCLIC GROUPS OF COMPOSITE ORDER

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Abstract: In this paper, first we have discussed about the basic concepts of Fuzzy and Intuitionistic Fuzzy Subgroups and have reviewed some results related to them. Then we state the main result - that we are trying to extend to Intuitionistic Fuzzy context - which states that, the level subgroups of a Fuzzy Subgroup of any finite group will form a chain. In one of our previous works we have disproved it in the case of acyclic groups with the help of a counter example. In this paper we prove that the chain structure fails in the case of finite cyclic groups of composite order through some counter examples and we arrive at a property that, if the composite number is a product of two distinct primes, then except in two of the possible Intuitionistic Fuzzy subgroups, the Intuitionistic Level Subgroups form a chain.

Key Words: Intuitionistic Level Subgroups (ILSG's), Chain, Finite cyclic group, Intuitionistic Fuzzy Subgroups (IFSG's).

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1. Introduction

In 1965 L. A. Zadeh introduced the concept of Fuzzy Set in his work *Fuzzy Sets.* The introduction of the concept of fuzzy sets, created a great impulse in generalising many of the abstract mathematical structures to fuzzy context. As a consequence, the concepts of fuzzy groupoids and fuzzy subgroups were introduced in 1971 by A Rosenfeld [9]. Later, K. T. Atanassov introduced the notion of Intuitionistic Fuzzy Set (IFS) in 1983 in his works [1] and [3]. In 1996 R Biswas came up with the notion of IFSG[4] as a generalization of Rosenfeld's definition of Fuzzy Subgroups. Many researches are still going on in establishing fuzzy and Intuitionistic fuzzy analogues of the results in group theory.

In this paper, first we discuss some basic concepts of Fuzzy and Intuitionistic Fuzzy Subgroups and review some results related to them. Then we review the result which states that, the level subgroups of a Fuzzy Subgroup of any finite group will form a chain. Our work is based on whether this result can be generalised to intuitionistic fuzzy context. In one of our previous works we have disproved it in the case of acyclic groups with the help of a counter example. In this paper we prove that the chain structure fails in the case of finite cyclic groups of composite order through some counter examples and we arrive at a property that if the composite number is a product of two distinct primes, then except in two of the possible Intuitionistic Fuzzy subgroups, the Intuitionistic Level Subgroups form a chain.

2. Preliminary Concepts

Here we discuss some basic definitions and results about Fuzzy Subgroups (FSGs) and Intuitionistic Fuzzy Subgroups (IFSGs). Throughout the paper, G

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denotes a multiplicative group, unless otherwise stated. Also $\wedge$ and $\vee$ denote the "min" and "max" operators on I.

2.1. Definition. A ***Fuzzy Subset*** A of a set X is characterised by a membership function $A : X \to [0, 1]$ which associates with each element in X a real number in the interval [0, 1], with the value $A(x)$ representing the "grade of membership" of x in A.

2.2. Definition. Let G be any group and A be any Fuzzy Subset of G. Then A is said to be a ***Fuzzy Subgroup*** (FSG) of G if $\forall\, x, y \in G$,

(1) $A(xy) \geq \wedge\{A(x), A(y)\}$ and

(2) $A(x^{-1}) = A(x)$.

2.3.Proposition. Let G be a group with identity element e and A be a FSG of G. Then $A(e) \geq A(x), \forall\, x \in G$.

2.4.Definition. Let X be any set, A be a fuzzy subset of a set X and $t \in I$. Then ***level subset*** of A at t (or t-cut of A) denoted by A_t is defined as $A_t = \{x \in X: A(x) \geq t\}$.

2.5. Proposition. Let G be a group with identity element e and A be a Fuzzy Subset of G. Then, A is a FSG of G, if and only if, A_t is a subgroup of G, $\forall\, 0 \leq t \leq A(e)$ ■

2.6. Definition. Let G be a group with identity element e and A be a FSG of G. Then the crisp subgroup A_t of G is called ***level subgroup*** of A at t, $\forall\, 0 \leq t \leq A(e)$.

2.7. Notation. If G is a group with identity element e and A is an FSG of G, then by G_A we denote the set $\{g \in G: A(g) = A(e)\}$.

2. 8. Proposition. Let G be a finite group and A be a FSG of $G$with $Im(A) = \{t_i: i = 1, 2, 3, \dots, n\}$. Then the collection $\{A_{t_i}: i = 1, 2, \dots, n\}$ contains all level subgroups of A.

Furthermore, if $t_1 > t_2 > \dots > t_n$, then all these level subgroups will form a chain $G_A = A_{t_1} \subseteq A_{t_2} \subseteq \dots \subseteq A_{t_n} = G$ ■

3. Intuitionistic Fuzzy Subsets and Intuitionistic Level Subgroups.

3.1. Definition. An ***Intuitionistic Fuzzy Subset*** (IFS) of a set X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A, \nu_A : X \to I$ represent the *grade of membership* and *grade of non membership* of any element $x \in X$ and should satisfy the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall\, x \in X$.

3.2. Definition. Let G be any group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFS of G. Then A is said to be an ***Intuitionistic Fuzzy Subgroup*** (IFSG) of G iff

(1) $\mu_A(xy) \geq \wedge\{\mu_A(x), \mu_A(y)\}$

(2) $\mu_A(x^{-1}) = \mu_A(x)$

(3) $\nu_A(xy) \leq \vee\{\nu_A(x), \nu_A(y)\}$, and

(4) $\nu_A(x^{-1}) = \nu_A(x)$.

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3.3. Proposition. Let G be a group with identity element e and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G. Then, $\mu_A(e) \geq \mu_A(x)$and $\nu_A(e) \leq \nu_A(x), \forall\, x \in G$.

3.4. Definition. Let X be any set, $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an IFS of X and $\alpha, \beta \in I$ be such that $0 \leq \alpha + \beta \leq 1$. Then the ***Intuitionistic Level Subset*** (ILS) of A at (α, β) (or $(\alpha, \beta)-$ cut of IFS A) is the crisp set $A_{\alpha,\beta} = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$.

3.5. Proposition. Let G be any group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFS of G. Then A is an IFSG of $G \Leftrightarrow A_{\alpha,\beta}$ is a subgroup of $G\ \forall\, 0 \leq \alpha \leq \mu_A(e)\ \&\nu_A(e) \leq \beta \leq 1$ with $\alpha + \beta \leq 1$■

3.6. Definition. Let G be any group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G. Then the crisp subgroup $A_{\alpha,\beta}[0 \leq \alpha \leq \mu_A(e)\ \&\nu_A(e) \leq \beta \leq 1$ with $\alpha + \beta \leq 1]$ of G is called ***Intuitionistic Level Subgroup*** (ILSG) of A at (α, β).

3.7. Proposition. Let G be any finite group and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G with $Im(\mu_A) = \{t_i : i = 1, 2, \dots, n\}$ and $Im(\nu_A) = \{s_j : j = 1, 2, \dots, m\}$. Then the collection $\{A_{t_i, s_j} : i = 1, 2, \dots, n;\ j = 1, 2, \dots, m\}$ will contain all ILSG's of G■

3.8. Remark. The second part of proposition 2.8 will not hold true for IFSG. That is, even if $t_1 > t_2 > \dots > t_n$ and $s_1 < s_2 < \dots < s_m$ $(n < m)$, the ILSG's need not form a chain.

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The following is a counter example which proves this claim.

3.9. Example. Consider the Klein 4 – group $G = \{e, a, b, ab\}$ where e is the identity element, $a^{-1} = a$, $b^{-1} = b$and $ab = ba$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in G\}$ be an IFSG of G, where μ_A and ν_A are defined by:

$$\mu_A : e \mapsto 1, a \mapsto 1/2, b \mapsto 1/3, ab \mapsto 1/3$$
$$\nu_A : e \mapsto 0, a \mapsto 1/4, b \mapsto 1/5, ab \mapsto 1/4$$

Then the ILSG's of A are $A_{1,0} = \{e\}$, $A_{1/2,1/4} = \{e, a\}$, $A_{1/3,1/5} = \{e, b\}$, $A_{1/3,1/4} = G$ which does not form a chain.

4. Extension Of The Result To IFSG's Of Finite Cyclic Groups Of Composite Order

The discussion in the previous section proves that ILSG's in a finite non-cyclic group G need not form a chain. In this paper we check whether the chain structure holds true when G is a finite cyclic group of composite order.

4.1. Proposition. In any IFSG of a finite cyclic group, the generators will have the minimum membership value and maximum non-membership value■

4.2. Remark. Since every finite cyclic group is isomorphic to $(\mathbb{Z}_n, +_n)$, in this paper we will be using groups of this kind where n is a composite number.

4.3. Example. Consider $n = 6$. Any cyclic group of order $n = 6$ will be isomorphic to $(\mathbb{Z}_6, +_6)$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbb{Z}_6\}$ be an IFSG of $\mathbb{Z}_6$.

We will now find out all the possible definitions of μ_A and ν_A satisfying all the axioms of IFSG's.

Since 0 is the identity element, it should have the maximum membership degree and the minimum non-membership degree. Also, since 1 and 5 are generators, they should have the minimum membership degree and maximum non-membership degree. 2 and 4 should have the same membership and non-membership degrees being the inverses of each other. Thus we get:

$\mu_A(1) = \mu_A(5)$& $\mu_A(2) = \mu_A(4)$ and $\mu_A(0) \geq \mu_A(2), \mu_A(3) \geq \mu_A(1)$

$\upsilon_A(1) = \upsilon_A(5)$& $\upsilon_A(2) = \upsilon_A(4)$and $\upsilon_A(0) \leq \upsilon_A(2), \upsilon_A(3) \leq \upsilon_A(1)$.

Next we need to find the relationship between $\mu_A(2), \mu_A(3)$and $\upsilon_A(2), \upsilon_A(3)$.

Since $1 = 3 +_6 4$, from the axioms of IFSG we get:

$$\mu_A(2) > \mu_A(3) \Rightarrow \mu_A(3) = \mu_A(1)$$

$$\mu_A(2) < \mu_A(3) \Rightarrow \mu_A(2) = \mu_A(1)$$

$$\mu_A(2) = \mu_A(3) \Rightarrow \mu_A(3) = \mu_A(2) = \mu_A(1)$$

Hence the membership degree μ_A can be defined in three different ways:

$$\mu_A(0) \geq \mu_A(2) > \mu_A(3) = \mu_A(1)$$

OR

$$\mu_A(0) \geq \mu_A(3) > \mu_A(2) = \mu_A(1)$$

OR

$$\mu_A(0) \geq \mu_A(2) = \mu_A(3) = \mu_A(1)$$

Similar reasoning holds true for non-membership degrees also. Hence the non-membership degree υ_A can also be defined in three different ways:

$$\upsilon_A(0) \leq \upsilon_A(2) < \upsilon_A(3) = \upsilon_A(1)$$

OR

$$\upsilon_A(0) \leq \upsilon_A(3) < \upsilon_A(2) = \upsilon_A(1)$$

OR

$$\upsilon_A(0) \leq \upsilon_A(2) = \upsilon_A(3) = \upsilon_A(1)$$

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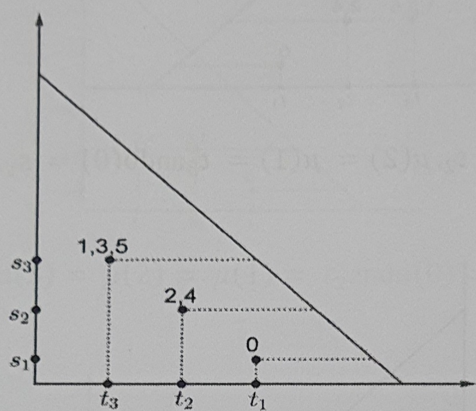
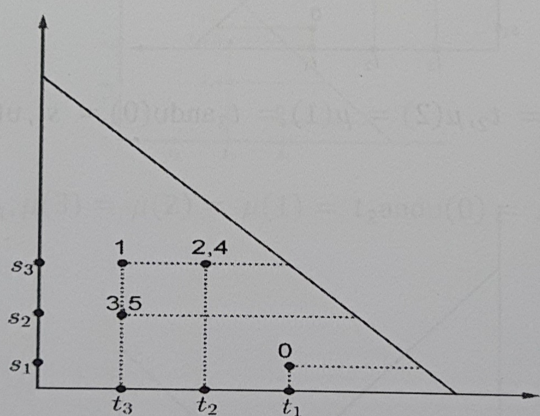
Hence there are 9 distinct IFSG's possible for $\mathbb{Z}_6$. We may call them $A_1, A_2, \dots, A_9$.

Let $s_1, s_2, s_3, t_1, t_2, t_3 \in [0,1]$ be such that $s_1 < s_2 < s_3, t_1 > t_2 > t_3 \& s_i + t_j \leq 1$.

$\boldsymbol{A_1}$: $\mu(0) = t_1, \mu(2) = t_2, \mu(3) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(2) = s_2, \upsilon(3) = \upsilon(1) = s_3$



$\boldsymbol{A_2}$: $\mu(0) = t_1, \mu(2) = t_2, \mu(3) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(3) = s_2, \upsilon(2) = \upsilon(1) = s_3$



$\boldsymbol{A_3}$: $\mu(0) = t_1, \mu(2) = t_2, \mu(3) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(3) = \upsilon(2) = \upsilon(1) = s_2$

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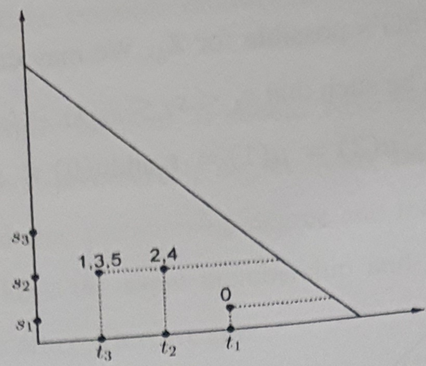
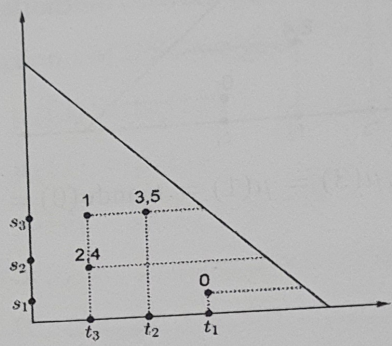
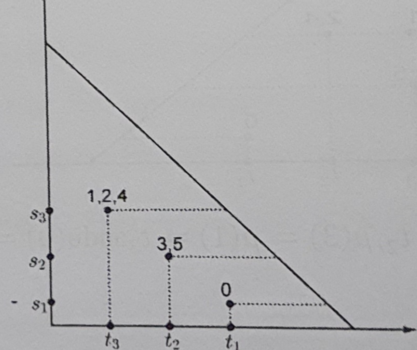


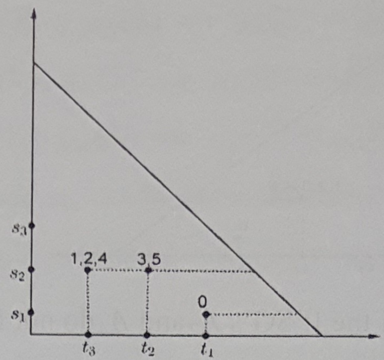
$\boldsymbol{A_4}$: $\mu(0) = t_1, \mu(3) = t_2, \mu(2) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(2) = s_2, \upsilon(3) = \upsilon(1) = s_3$



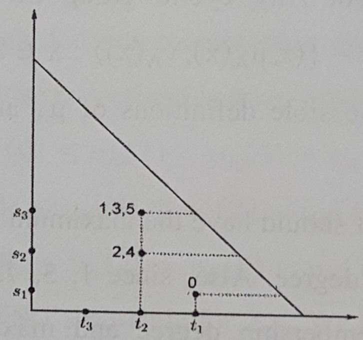
$\boldsymbol{A_5}$: $\mu(0) = t_1, \mu(3) = t_2, \mu(2) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(3) = s_2, \upsilon(2) = \upsilon(1) = s_3$



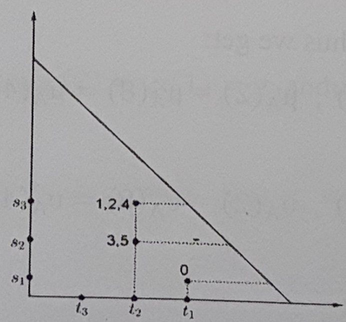
A_6: $\mu(0) = t_1, \mu(3) = t_2, \mu(2) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(3) = \upsilon(2) = \upsilon(1) = s_2$



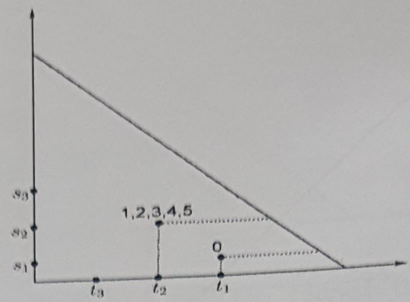
A_7: $\mu(0) = t_1, \mu(3) = \mu(2) = \mu(1) = t_2$ and $\upsilon(0) = s_1, \upsilon(2) = s_2, \upsilon(3) = \upsilon(1) = s_3$



A_8: $\mu(0) = t_1, \mu(3) = \mu(2) = \mu(1) = t_2$ and $\upsilon(0) = s_1, \upsilon(3) = s_2, \upsilon(2) = \upsilon(1) = s_3$



A_9: $\mu(0) = t_1, \mu(3) = \mu(2) = \mu(1) = t_2$ and $\upsilon(0) = s_1, \upsilon(3) = \upsilon(2) = \upsilon(1) = s_2$



It can be seen that the ILSG's of the IFSG's A_2 and A_4 do not form a chain. Hence the chain structure of ILSG's may not always hold true for finite cyclic groups of composite order.

4.4. Example. Consider $n = 10$. Any cyclic group of order $n = 10$ will be isomorphic to $(\mathbb{Z}_{10}, +_{10})$. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in \mathbb{Z}_6\}$ be an IFSG of $\mathbb{Z}_{10}$. We will now find out all the possible definitions of μ_A and ν_A satisfying all the axioms of IFSG's.

Since 0 is the identity element, it should have the maximum membership degree and the minimum non-membership degree. Also, since 1, 3, 7, 9 are generators, they should have the minimum membership degree and maximum non-membership degree. 2, 8 and 4, 6 should have the same membership and non-membership degrees being the inverses of each other. Since $4 = 2 +_{10} 2$, $2 = 6 +_{10} 6$ and from the axioms of IFSG we get $\mu_A(2) = \mu_A(4)$. Thus we get:

$\mu_A(1) = \mu_A(3) = \mu_A(7) = \mu_A(9)$, $\mu_A(2) = \mu_A(8) = \mu_A(4) = \mu_A(6)$ and $\mu_A(0) \geq \mu_A(2), \mu_A(5) \geq \mu_A(1)$

$\upsilon_A(1) = \upsilon_A(3) = \upsilon_A(7) = \upsilon_A(9)$, $\upsilon_A(2) = \upsilon_A(8) = \upsilon_A(4) = \upsilon_A(6)$ and $\upsilon_A(0) \leq \upsilon_A(2), \upsilon_A(5) \leq \upsilon_A(1)$

Next we need to find the relationship between $\mu_A(2), \mu_A(5)$ and $\upsilon_A(2), \upsilon_A(5)$.

Since $1 = 5 +_{10} 6$, from the axioms of IFSG we get:

$$\mu_A(2) > \mu_A(5) \Rightarrow \mu_A(5) = \mu_A(1)$$
$$\mu_A(2) < \mu_A(5) \Rightarrow \mu_A(2) = \mu_A(1)$$
$$\mu_A(2) = \mu_A(5) \Rightarrow \mu_A(1) = \mu_A(2) = \mu_A(5)$$

Hence the membership degree μ_A can be defined in three different ways:

$$\mu_A(0) \geq \mu_A(2) > \mu_A(5) = \mu_A(1)$$

OR

$$\mu_A(0) \geq \mu_A(5) > \mu_A(2) = \mu_A(1)$$

OR

$$\mu_A(0) \geq \mu_A(2) = \mu_A(5) = \mu_A(1)$$

Similar reasoning holds true for non-membership degrees also. Hence the non-membership degree υ_A can also be defined in three different ways:

$$\upsilon_A(0) \leq \upsilon_A(2) < \upsilon_A(5) = \upsilon_A(1)$$

OR

$$\upsilon_A(0) \leq \upsilon_A(5) < \upsilon_A(2) = \upsilon_A(1)$$

OR

$$\upsilon_A(0) \leq \upsilon_A(2) = \upsilon_A(5) = \upsilon_A(1)$$

Hence there are 9 distinct IFSG's possible for $\mathbb{Z}_{10}$. We may call them $A_1, A_2, \dots, A_9$. Among these, the chain structure of ILSG's holds true in all the IFSG's except

A_1: $\mu(0) = t_1, \mu(2) = t_2, \mu(5) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(5) = s_2, \upsilon(2) = \upsilon(1) = s_3$ and

A_2: $\mu(0) = t_1, \mu(5) = t_2, \mu(2) = \mu(1) = t_3$ and $\upsilon(0) = s_1, \upsilon(2) = s_2, \upsilon(5) = \upsilon(1) = s_3$

where $s_1, s_2, s_3, t_1, t_2, t_3 \in [0,1]$ are such that $s_1 < s_2 < s_3, t_1 > t_2 > t_3$ & $s_i + t_j \leq 1$.

5. Conclusion

In this paper we have reviewed a counter example which disproves the chain structure of ILSG's in a non-cyclic group. Also we have proved that the chain structure fails in the case of finite cyclic groups of composite order through some counter examples and have arrived at a property that if the composite number is a product of two distinct primes, then except in **two** of the possible Intuitionistic Fuzzy subgroups, the Intuitionistic Level Subgroups form a chain. In the upcoming paper we will try to prove this property rigorously and will also consider the other possibilities of the composite order.

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