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A STUDY OF SOME INDUCED FUNCTIONS BETWEEN THE LATTICES OF FUZZY SETS

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Abstract: In this paper, we analyse the properties of three induced functions between the lattices of fuzzy sets. These are induced by a pair of functions, one between the underlying sets and the other between the underlying membership sets. The study is further extended to the case of lattices of fuzzy topologies.

1. Introduction

A function between two sets induces two functions between their power sets, in a natural way. Properties of the induced functions depend greatly on the properties of the inducing function. A similar study on the case of fuzzy sets is interesting and more so by varying the membership sets. Different order structures for membership sets were considered by J.A Goguen [5] and he suggested that a complete and distributive lattice would be a minimum structure for the membership set. Many mathematicians have used different lattice structures for developing fuzzy set theory, like (i) completely distributive lattice with 0 and 1 by T.E.Gantner, E.Steinlage and R.H.Warren [4], (ii) complete and completely distributive lattice with order reversing involution by Bruce Hutton and Ivan Reilly [2]; (iii) complete and completely distributive non-atomic boolean algebra by Mira sarkar [6] etc. We take complete and completely distributive lattices as the membership sets. S.E.Rodabough [7] had used induced functions to define morphisms in the category of L-fuzzy topologies: FUZZ. We take the definition of a fuzzy topology as given by C.L.Chang [3].

2. Preliminaries

Let X be an arbitrary set and L be a complete and completely distributive lattice. A function $a : X \to L$ is called a ***fuzzy subset*** of X. The collection of all fuzzy sets on X is denoted by L(X). It is a complete and distributive lattice [I]. A subset T of L(X) containing the smallest and largest fuzzy subsets of X, closed for finite meet operations, and arbitrary join operations among its members, is a ***fuzzy topology*** on X. The collection of all fuzzy topologies on X is denoted by (L,X). It is a complete lattice with {0, 1} as the smallest element and L(X) as the largest element [1]. '0' and '1' are commonly used to denote the smallest and largest elements, respectively in a lattice.

3. Definitions of induced functions

Let X, Y be sets and L, M be complete and completely distributive lattices. Let $f : X \to Y$, $g : L \to M$ and $h : M \to L$ be given functions. Define the ***induced functions*** $E : L(X) \to M(Y)$, $F: M(Y) \to L(X)$ and $H: M(Y) \to L(X)$ as follows:

For $a \in L(X)$ and $y \in Y$, $E(a)(y) = g(\vee(a(f^{-1}(y))))$;

For $b \in M(y)$ and $x \in X$, $F(b)(x) = \vee(g^{-1}(b(f(x))))$ and

For $d \in M(Y)$ and $x \in X$, $H(d)(x) = h(d(f(x)))$.

Let A be a subset of X. Then the ***characteristic function*** on A, is a fuzzy subset defined by

$$\mathrm{char}(A)(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

4. Properties of E

4.1. Theorem. *(i) E is one to one iff f and g are one to one, (ii) E is onto iff f and g are onto i and hence E is a bijection iff f and g are bijections.*

Proof. Straight forward. □

4.2. Theorem. *If g is a non-constant function, then E is a lattice homomorphism iff g is a homomorphism and f is one to one.*

Proof. ***Necessity*:** Suppose g is not a homomorphism, then there exist l, m in L such that either

$$g(l \vee m) \neq g(l) \vee g(m) \text{ or } g(l \wedge m) \neq g(l) \wedge g(m).$$

Correspondingly for the constant fuzzy subsets $\underline{l}$ and $\underline{m}$

$$\begin{array}{ll} \text{either} & E(\underline{l} \vee \underline{m}) \neq E(\underline{l}) \vee E(\underline{m}) \\ \text{or} & E(\underline{l} \vee \underline{m}) \neq E(\underline{l}) \wedge E(\underline{m}). \end{array}$$

Therefore, for E to be a lattice homomorphism, g must be a lattice homomorphism.

Suppose g is a non-constant function, E is a lattice homomorphism, and f is not one to one, then there exist w and x in X such that $w \neq x$ and $f(w) = f(x) = z$ (say). Let $a = \mathrm{char}(\{w\})$ and $b = \mathrm{char}(\{x\})$. Then $a \wedge b = 0$ in L(X), but $E(a \wedge b)(z) = g(\vee(a\wedge b)f^{-1}(z)) = g(0)$, whereas $E(a)(z) = E(b)(z) = g(1)$. Since for a non-constant lattice homomorphism g, $g(0) \neq g(1)$, we have that $E(a \wedge b) \neq E(a) \wedge E(b)$. Therefore, E is not a lattice homomorphism. Thus f must be onto.

***Sufficiency*:** Let f be one to one and g be a lattice homomorphism. Then for a,b in L(X) and y in Y

$$E\,(aVb)\,(y) = g\,(V\,(aVb)\,f^{1}\,(y))$$
$$= g((af^{1}(Y(b(f^{1}(y))) \text{ , since f is one to one}$$
$$= g(af^{1}\,(y)\,)Vg.(bf^{1}(y)), \text{ since g is homomorphism}$$
$$= E\,(a)\,(y)VE\,(b)\,(y)$$

Similarly, E (a Λ b) (y) = E (a) (y) Λ E (b) (y) . Hence E is a lattice homomorphism. □

4.3. Note. If g is a constant function then E is a lattice homomorphism irrespective of whether f is one to one or not, since E, in this case will also be a constant function.

4.4. Definition. Let J and K be two complete lattices and h:J→ K be a function such that

(i) h is a homomorphism,
(ii) h(0) = 0 , h(l) = 1, and
(iii) h (Vl (i)) = Vh (1 (i)) where {l (i) | i in I} is an arbitrary subset of J.

Then h is called a ***t-homomorphism***. "t" in the above definition is indicative of the fact that a t-homomorphism takes a fuzzy topology to a fuzzy topology. In particular when J = K = 2(X), a t-homomorphism takes a topology to another topology on X .

4.5. Theorem. *E is a homomorphism with E(0) = 0 and E(l) = 1 iff g is a homomorphism, g(0) = 0 and g(l) = 1, and f is a bijection.*

Proof. Straight forward. □

4.6. Theorem. *Let f be a bijection. Then E is a t-homomorphism iff g is a t-homomorphism.*

Proof. In the light of the theorem 4.5, to complete the proof, it is enough to show that E preserves arbitrary join operation iff g does so.

Necessity: Suppose g does not preserve arbitrary join operation, then there exists {l(i) | i in I}, an arbitrary subset of L such that $g(\vee l(i)) \neq \vee g(l(i))$. Consider the constant fuzzy subsets $\underline{l}(i)$, for each i. We then have that $E(\vee\underline{l}(i)) \neq \vee E(\underline{l}(i))$. ie., E does not preserve arbitrary join operation.

Sufficiency: Let g preserve arbitrary join operation and {a(i) | i in I} be an arbitrary subset of L(X). Then for each y in Y,

$$\begin{aligned} E(\vee a(i))(y) &= g(\vee(\vee a(i)f^{-1}(y)) \\ &= g(\vee(a(i)f^{-1}(y)))\,, \quad \because\ f \text{ is a bijection} \\ &= \vee g(a(i)f^{-1}(y)), \because g \text{ preserves arbitrary join operation} \\ &= \vee E(a(i))(y) \\ \therefore E(\vee a(i)) &= \vee E(a(i)).\ \square \end{aligned}$$

4.7. Remark. Taking L = M and g to be the identity function, we have that g is a t-homomorphism, which is also a bijection. In the light of the above theorem, we get, corresponding to every function $f:X\rightarrow Y$, there exists a function $E:L(X)\rightarrow L(Y)$ such that

(i) E is one to one iff f is one to one
(ii) E is onto iff f is onto , and
(iii) E is an onto t-isomorphism iff f is a bijection.

Taking X = Y and f to be the identity function, there exists a naturally induced function $E:L(X)\rightarrow M(X)$, corresponding to every function $g:L\rightarrow M$ such that

(i) E is one to one iff g is one to one
(ii) E is onto iff g is onto ,and
(iii) E is a t-homomorphism iff g is a t-homomorphism.

4.8. Definition. A t-homomorphism $E:L(X)\rightarrow M(Y)$ induces a function $E':(L, X) \rightarrow (M, Y)$ where $E'(T) = \{E(t) \mid t \in T\}$ for each T in (L, X) . Clearly E'(T) is a fuzzy topology on Y, for each T in (L,X).

4.9. Remark. The following observations on E' are immediate:

(i) $E'(0) = 0$

(ii) E' (1) = 1 iff E is onto, and
(iii) E' (VT(i)) = V(E' (T(i)) for T(i) in (L,X) and i in I.

The proofs of the following two theorems are omitted since they are straight forward.

4.10. Theorem. *Let $E : L(X) \to M(Y)$ be a t-homomorphism and E' : $(L,X) \to (M,Y)$ be the induced function. Then (i) E' is one to one iff E is one to one (ii) E' is onto iff E is onto (iii) E' is a homomorphism iff $E^{-1}(d)$ is either a singleton or empty for all d in M(Y) and $d \neq 0,1$ (iv) E' is an onto t-isomorphism iff E is a bijection.* □

4.11. Theorem. *If $f : X \to Y$ is a bijection, then the following are equivalent, for the induced functions. (i) $g:L \to M$ is an onto t-isomorphism, (ii) $E:L(X) \to M(Y)$ is an onto t-isomorphism, and (iii) $E': (L,X) \to (M,Y)$ is an onto t-isomorphism.* □

5. Properties of F

5.1. Theorem. *(i) If g is a bijection, then F is one to one iff f is onto (ii) F is one to one does not imply either g is one to one or g is onto.But, if F is one to one, then g is one to one implies g is onto and (iii) F is one to one implies f is onto.*

Proof. straight forward. □

5.2. Example. Let L = M = X = Y =[0,1] and $f:X \to Y$ be the identity function. Let {p (n) | n = 1,2,...} be a set of distinct prime numbers. Let A(n) denote the set of all p(n)-adic rationals in (0,1) and let {q(n) | n = 1,2,...} be the set of rationals in (0, 1) . Now define for each x in [0,1], subsets:

$$B(x) = \begin{cases} x & \text{, if x is a irrational} \\ 0 & \text{, if } x = 0 \\ A(n) \cap [0, x] & \text{, if } x = q(n) \\ (U\{A(n)- B(q(n))\})\ U\ \{1\} & \text{, if } x = 1 \end{cases}$$

Now {B (x) | x ∈ [0,1]} is a partition of [0,1] such that VB(x) = x, for every x in X. Define g:L →M such that for each l in L, g (l) = x if $l \in B(x)$. Then g is onto, but not one to one. However for a,b in M(Y), F(a) = F(b) implies a = b since, F (a) = F (b) implies F (a) (x) = F (b) (x), for each x in X.

$$\text{ie. } Vg^{-1}af(x) = Vg^{-1}bf(x)$$

$$\text{ie. } Vg^{-1}a(x) = Vg^{-1}b(x), \quad \text{since f is the identity.}$$

But this implies that a (x) = b (x) for each x. Thus F is one to one, though g is not one to one. Moreover, here F is a bijection (identity function).

5.3. Example. Let L = M = [0,1] and g:L →M be the constant function $\underline{1}$. Let X and Y be two sets such that there exists an onto function f :X→Y. Now if for some a,b in M(Y), F(a) = F(b) then F(a) (x) = F(b) (x) for all x in X.

$$\text{ie. } Vg^{-1}af(x) = Vg^{-1}bf(x)$$

Thus we have $Vg^{-1}af(x) = Vg^{-1}bf(x) = 0$ or 1 for each x. $Vg^{-1}af(x) = Vg^{-1}bf(x) = 0$ implies af(x) = bf(x) = 0 and $Vg^{-1}af(x) = Vg^{-1}bf(x) = 1$ implies af (x) = hf (x) = 1 for $x \in x$. Since f is onto, a = b. ie., F is one to one, though g is not onto. □

5.4. Definition. Let h : L → M be a function and L/h denote the set $\{A(m)=h^{-1}(m) \mid m \in M\}$. A(m) is called the ***fiber*** of h at m. Define V and Λ operation in L/h as follows: for $m, n \in M$

$$A(m) \vee A(n) = A(m \vee n) \text{ and } A(m) \wedge A(n) = A(m \wedge n).$$

Then L/h becomes a lattice which is complete and completely distributive with least element A(0) and the largest element A(l). A function J:L/h→L defined by J(A(m)) = V A(m) is called the ***join function*** on the fibers of h. The fibers of h are distinct iff h is either onto or M-h (L) is a singleton.

5.5. Theorem. *F is one to one iff (i) f is onto, (ii) the fibers of g are distinct, and (iii) the join function on the fibers of g is one to one.*

Proof. ***Necessity*:** f must be onto follows from theorem 5.1. Suppose the fibers are not distinct then there exist m,n in M such that $m \neq n$ and $g^{-1}(m) = g^{-1}(n)$. Consider the constant fuzzy subsets $\underline{m}$ and $\underline{n}$ in M(Y), we have $\underline{m} \neq \underline{n}$

but F($\underline{m}$) = F($\underline{n}$). Thus F is not one to one, a contradiction. Therefore, the fibers of g must be distinct.

Suppose the join function on the fibers of g is not one to one, then there exist m,n in M such that $\vee g^{-1}(m) = \vee g^{-1}(n)$ but $m \neq n$. Thus the constant fuzzy subsets $\underline{m}$ and $\underline{n}$ are different but F($\underline{m}$) = F($\underline{n}$), a contradiction to the assumption that F is one to one. Hence the join function on the fibers of g must be one to one. □

Sufficiency: Let F(c) = F(d) for some c,d in M(Y). Then F (c) (x) = F (d) (x) for all x in X. ie., $\vee g^{-1}$ cf(x) = $\vee g^{-1}$ df(x) for each x in X. This implies cf(x) = df(x) for each x, since join function on the fibers of g is one to one. Thus c = d. Since c and d are arbitrary, F is one to one.

The proofs of the following four theorems are straight forward and hence are omitted. □

5.6. Theorem. *(i) If g is one to one then F is onto iff f is one to one. (ii) F is onto, does not imply that g is one to one. (iii) f is one to one does not imply that F is onto. (iv) F is onto iff f is one to one and the join function on the fibers of g is one to one.* □

5.7. Theorem. *(i) g is an onto isomorphism, implies F is a homomorphism (ii) F is a homomorphism, does not imply that (a) g is one to one, (b) g is a homomorphism, and (c) g is onto. (iii) F is a homomorphism iff join function on the fibers of g is a homomorphism.* □

5.8. Theorem. *If $g^{-1}(0) \subseteq \{0\}$ and the join function on the fibers of g is a t-homomorphism, then F is at-homomorphism.* □

5.9. Theorem. *(i) $F(0) = 0$ iff $g^{-1}(0) \subseteq \{0\}$ (ii) g is a t-homomorphism and $g^{-1}(0) \subseteq \{0\}$, do not imply that F is a homomorphism.* □

5.10. Remark. If g is a one to one and onto lattice homomorphism then F is an onto t-homomorphism. But the converse is not true. This follows from example 5.2, where F is an onto t-isomorphism, while g is neither one to one nor a lattice homomorphism.

5.11. Theorem. *$F \bullet E$ is the identity function iff f and g are one to one.*

Proof. ***Necessity*:** Suppose f is not one to one, then there exist w,x in X such that w ≠ x and f(w) = fc{x}. Let a be the fuzzy subset of X defined by, for y in X,

$$a(y) = \begin{cases} 0 & \text{if } y = w \\ 1 & \text{if } y \neq w \end{cases}$$

Now

$$\begin{aligned}(F \bullet E)(a)(w) &= Vg^{-1}(g(Vaf^{-1}f(w))) \\ &= Vg^{-1}(g(Va(\{w,x\}))) \\ &= Vg^{-1}(g(1)) \\ &= 1 \neq a(w).\end{aligned}$$

Therefore, (F•E) (a) ≠ a. ie., F•E is not the identity function on L(X). Hence f must be one to one.

Suppose g is not one to one, then there exist an l in L such $Vg^{-1}(g(l)) \neq l$. Let $\underline{l}$ be the constant fuzzy subset of X, with the membership value l. Then (F•E) ($\underline{l}$) ≠ $\underline{l}$, and hence F•E is not the identity function. Thus g must be one to one.

Sufficiency: Let $a \in L(X)$. Then for each x in X,

$$\begin{aligned}(F \bullet E)(a)(x) &= Vg^{-1}(E(a)(x)) \\ &= Vg^{-1}(g(af^{-1}(f(x))) \\ &= Vg^{-1}g(a(x)) \quad , \because f \text{ is one to one} \\ &= a(x), \because g \text{ is one to one}\end{aligned}$$

Thus F•E is the identity function on L(X). □

5.12. Definition. A subset S of a complete lattice is said to be ***upper complete*** if VS belongs to S.

We omit the proofs of the remaining theorems in this section.

5.13. Theorem. *E • F is the identity function on M(Y) if and only if **(i)** f is onto, **(ii)** g is onto, and **(iii)** the fibers of g are upper complete.* □

5.14. Theorem. *If $f: X \to Y$ is a bijection, then the following are* *equivalent, for the induced functions:* ***(i)*** *g:$L \rightarrow M$ is an onto t-isomorphism.* ***(ii)*** *F:$M(Y) \rightarrow L(X)$ is an onto t-isomorphism.* ***(iii)*** *F^1:$(M,Y) \rightarrow (L,X)$ is an onto t-isomorphism.* □

6. Properties of H

In this section we state some theorems regarding properties of the induced function H. Their proofs are omitted.

6.1. Theorem. ***(i)*** *H is one to one iff f is opto and h is one to one, and* ***(ii)*** *H is onto iff f is one to one and h is onto.* □

6.2. Theorem. *H is lattice homomorphism if and only if h is so.* □

6.3. Theorem. *H is a t-homomorphism iff h is so.* □

6.4. Theorem. ***(i)*** *H is a t-isomorphism iff h is a t-isomorphism and f is onto.* ***(ii)*** *H is an onto t-isomorphism iff h is an onto t-isomorphism and f is a bijection.* □

6.5. Observation. Taking L = M and h:M→L to be the identity function, we have that, corresponding to every function f:X→Y, there exist a function H:M(Y)→L(X) such that

(i) H is one to one iff f is onto,
(ii) H is onto iff f is one to one, and
(iii) H is a t-homomorphism for any f.

Taking X = Y and f:X→Y to be the identity function, we get corresponding to every function h:M→L, there exists a function H:M(Y)→L(X) such that

(i) H is one to one iff h is one to one,
(ii) H is onto iff h is onto,
(iii) H is a lattice homomorphism iff h is a lattice homomorphism,
and (iv) H is a t-homomorphism iff h is a t-homomorphism.

6.6. Note. The image of a fuzzy topology on Y, under a t-homomorphism $H : M(Y) \rightarrow L(X)$, is a fuzzy topology on X. Hence H in this case, induces a function $H^1: (M, Y) \rightarrow (L, X)$ such that $H^1(U) = \{H(u) \mid u \text{ in } u\}$, for U in (M, Y).

The following observations on H^1 are immediate.

(i) $H^1(0) = 0$ always,

(ii) $H^1(1) = 1$ iff H is onto, and

(iii) H^1 preserves arbitrary join operation.

6.7. **Theorem.** *If $f : X \rightarrow Y$ is a bijection, then the following are equivalent for the induced functions.* ***(i)*** *$h : M \rightarrow L$ is an onto t-isomorphism* ***(ii)*** *$H : M(Y) \rightarrow L(X)$ is an onto t-isomorphism* ***(iii)*** *$H^1 : (M, Y) \rightarrow (L, X)$ is an onto t-isomorphism.* □

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