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A STUDY OF POTENTIAL FUNCTIONS AND THEIR APPLICATIONS

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Abstract: In this paper potential.functions satisfying Laplaces equations expressed in series have been studuied with some applications, in wave motions.

1. Introduction

Let (α,β,γ) be a fixed point and (ξ,η,ζ) be any point in the neighbourhood of (α,β,γ). Then $x = \alpha+\xi$, $y = \beta+\eta$, $z = \gamma+\zeta$ where the point (x,y,z) is refered to O as origin. Let D be the domain of convergences of an infinite series of a potential function $V(x,y,z)$ and (α,β,γ) be any fixed point in *D*. *V* is analytic at every (α,β,γ) in D. It is known that every potential function is the sum of finite or infinite number of homogeneous functions of degrees 0,1,2,3,.... in (ξ,η,ζ) and so the ***Laplace's equation***

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0$$

is transformed into

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}+\frac{\partial^2 V}{\partial \zeta^2} = 0$$

which possesses 2n+1 number of independent solutions. Fourier series or Fourier function satisfies all sorts of Laplace's eqnations many of which are obtained by transformation processes.

In this paper our main aim is to discuss potential and its associated functions expressed in either series or integrals.

2. Pre requisites

We first introduce an analytic function $F_n(u)$ expressed by

$$F_n(u) = \frac{A_0(\xi,\eta,\zeta)}{2} + \sum_{m=1}^{n}\left\{A_m(\xi,\eta,\zeta)\cos mu + B_m(\xi,\eta,\zeta)\sin mu\right\} \quad \textbf{(1)}$$

with $\lim_{n\to\infty} F_n(u) = F(u)$. If a potential function be defined by

$F_n(\xi, \eta, \zeta, u) = (\zeta + i\xi \cos u + i\eta \sin u)^n$, (u is a variable parameter)
then it is easy to see that

$$\frac{\partial^2 \upsilon_n}{\partial \xi^2}+\frac{\partial^2 \upsilon_n}{\partial \eta^2}+\frac{\partial^2 \upsilon_n}{\partial \zeta^2} = n(n-1)\ (-\cos^2 u - \sin^2 u + 1)V_{n-2} = 0$$

where $F_n(\xi, \eta, \zeta)$ is a homogenous function of degree n in (ξ, η, ζ).

$$\left.\begin{aligned} \therefore \quad A_m(\xi,\eta,\zeta) &= {}^1/_\pi \int_{-\pi}^{\pi} F_n(\xi,\eta,\zeta\,;u)\cos mu\, du\,,\ (m\neq 0) \\ B_m(\xi,\eta,\zeta) &= {}^1/_\pi \int_{-\pi}^{\pi} F_n(\xi,\eta,\zeta\,;u)\sin mu\, du\,,\ (m\neq 0) \\ \text{and} \quad A_0(\xi,\eta,\zeta) &= {}^2/_\pi \int_{-\pi}^{\pi} F_n(\xi,\eta,\zeta\,;u)\, du \end{aligned}\right\} \quad \textbf{(2)}$$

So , F(u) in (1) with the values of $A_0(\xi,\eta,\zeta)$, $A_m(\xi,\eta,\zeta)$ and $B_m(\xi,\eta,\zeta)$ given in (2) is a ***Fourier function*** or the ***Fourier series*** (see[3]).

2.1. Proposition. *Let $F_1(u)$, $F_2(u)$, ……$F_{2n+1}(u)$ be the 2n+1 number of independent solutions of Laplace's equation. Then any other function $\varphi(u)$ that satfies Laplace's equation can be expressed as*

$$\varphi(u) = \Sigma \lambda_k F_k(u), \ (\lambda_k \neq 0,\ F_k(u) \neq 0) \ \square$$

2.2.Remark. Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$. Then using the polar coordinates r,θ,φ of a point (x,y,z), where $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, we have

$$F_n(x,y,z\,;u) = r^n(\cos\theta + i\sin\theta\cos\psi)^n, (\psi = u-\varphi)$$

$$\therefore \quad \pi A_m(x,y,z) = \int_{-\pi}^{\pi} V_n(x,y,z;u)\cos mu\,du$$

$$= r^n\cos n\varphi \int_{-\pi-\varphi}^{\pi-\varphi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m\psi\,d\psi$$

$$- r^n\sin m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} (\cos\theta + i\sin\theta\cos\psi)^n \sin m\psi\,d\psi$$

Similarly, $$\pi B_m(x,y,z) = \int_{-\pi}^{\pi} V_n(x,y,z;u)\sin mu\,du$$

$$= r^n\sin m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} (\cos\theta + i\sin\theta\cos\psi)^n \cos m\psi\,d\psi$$

$$+ r^n\cos m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} (\cos\theta + i\sin\theta\cos\psi)^n \sin m\psi\,d\psi$$

Let $$G_n(\theta,\psi) = (\cos\theta + i\sin\theta\cos\psi)^n$$

Then $$\pi A_m(x,y,z) = r^n\cos m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} G_n(\theta,\psi)\cos m\psi\,d\psi$$

$$-r^n\sin m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} G_n(\theta,\psi)\sin m\psi\,d\psi$$

and $$\pi B_m(x,y,z) = r^n\sin m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} G_n(\theta,\psi)\cos\psi\,d\psi$$

$$+ r^n\cos m\varphi \int_{-\pi-\varphi}^{\pi-\varphi} G_n(\theta,\psi)\sin m\psi\,d\psi$$

The difference between upper limit and lower limit of the integrals is 2π, that is, $(\pi-\varphi)-(-\pi-\varphi) = 2\pi$ which is independent of φ.

2.3. Proposition. *Let $g_n(u)$ be a rational function of $v = e^{iu}$. Then the potential function can be written as*

$$V(\xi,\eta,\zeta) = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} V_n(\xi,\eta,\zeta;u)\, g_n(u)\, du$$

Here, if we perform the summation first, then the series must converge to a function $H(\xi,\eta,\zeta;u)$ at (ξ,η,ζ) and so

$$V(\xi,\eta,\zeta) = \int_{-\pi}^{\pi} H(\xi,\eta,\zeta;u)\, du \quad \square$$

2.4. Proposition. *If $F_n(\xi,\eta,\zeta;u) = (\zeta + i\xi\cos u + i\eta\sin u,\ u)^n$ which is integrable over $[-\pi,\pi]$ with regard to u, then*

$$V^{(n)}(\xi,\eta,\zeta) = \int_{-\pi}^{\pi} F_n(\xi,\eta,\zeta;u)\, du$$

In particular,

$$V^{(1)}(\xi,\eta,\zeta) = \int_{-\pi}^{\pi} F_1(\xi,\eta,\zeta;u)\, du \quad \text{(see[3])} \quad \square$$

2.5. Remark. The functions defined by

$$L_n(x,y,z;u,\psi) = (z + ix\cos u + iy\sin u)^n \begin{matrix}\cos\\ \sin\end{matrix} m\psi$$

are even and odd functions of ψ respectively; also they are periodic functions of ψ. Legendre's functions are associated with such functions, and Ferrar defines them as

$$F_n(x,y,z;r,\theta,\varphi) = \int_{-\pi}^{\pi} (z + ix\cos u + iy\sin u)^n \begin{matrix}\cos\\ \sin\end{matrix}(mu)du$$

$$+ \frac{2\pi i^m \underline{|m}}{\underline{|n+m}}\, r^n P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi)$$

3. Solutions and Applications

3.1 Solutions in terms of Legendre's polynomials. It is seen that every solution of Laplace's equation which is analytic in the neighbourhood of the origin 0 is of the form

$$V(r,\theta,\varphi) = \sum_{n=0}^{\infty} r^n \left\{ A_n P_n(\cos\theta) + \sum_{m=1}^{n} \left[A_n^{(m)} \cos(m\varphi) + B_n^{(m)} \sin(m\varphi)\, P_n^{m}(\cos\theta) \right] \right\} \quad (3)$$

Let $S_n(\theta,\varphi)$, the bracketed expression in (3) be an analytic function of θ,φ where its form is stated. Then

$$V(r,\theta,\varphi) = \sum_{n=0}^{\infty} r^n S_n(\theta,\varphi)$$

The function $S_n(\theta,\varphi)$ is called a ***surface harmonic of degree*** n, and $r^n S_n(\theta,\varphi)$ is called a ***solid harmonic*** or ***sperical harmonic***. On the surface of a unit sphere there exist curves on which the functions ${}^{\cos}_{\sin}(m\varphi)P_n^{m}(\cos\theta)$ vanish are called ***tesseral harmonics***.

3.2 Periods of vibration of a uniform membrane. Vibration of a uniform membrane is a particular case of study of the most general wave equation for any kind of wave motion given by

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{C^2}\frac{\partial^2 v}{\partial t^2} \quad (4)$$

where v is the potential of a prticle at (x,y,z) at time t and the constant C possesses different meanings in different fields of study. The equation (4) is used in almost all natural sciences, astronomical sciences, electromagnetic and some other sciences.

Again if (x,y) is a point on a uniform membrane vibrating horizontally, and v is the displacement of the point at time t, then

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{C^2}\frac{\partial^2 v}{\partial t^2}$$

Putting z = ict , we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{C^2}\frac{\partial^2 v}{\partial (z/ic)^2} = -\frac{\partial^2 v}{\partial z^2}$$

or

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

which is the well known Laplace's equation.

In the case of a vibrating drum with circular end tightly stretched, V=0 at the boundary of the circular end . Here one possible type of vibration is given by the equation

$$V = \mathcal{J}_m (K_\rho) \cos(m\varphi) \cos(ckt)$$

where ρ is the radius of a concentric circle on the drum, R is the radius of the circular end and K is some suitable constant. The radius ρ = R is along the boundary circle, where

$$\mathcal{J}_m(K_R) = 0$$

There exists a most suitable value of K, say $K = K_r$ (r = 1,2,3……) for which the solution is

$$V = \mathcal{J}_m(K_r , \rho) \cos(m\varphi) \cos(CK_r t)$$

which is a periodic motion of period

$$T_r = 2\pi/ (CK_r), (r = 1,2,3,\ldots.)$$

where the calculation of these periods depend upon calculating the zeros of Bessel's co-efficients (see[3]).

3.3. Disturbance function. The disturbance function and the potential of the equation of general motion are defined by

$$D(x,y,z,u,v,t) = f (x \sin u \cos v + y \sin u \sin v + z \cos u + ct,u,v)$$

and

$$V = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} D(x,y,z,u,v,t)\, du\, dv$$

Also, referring to Bessel's co-efficient $\mathcal{J}_m$ or $\mathcal{J}_n$, the Bessel's function expressed in series is stated by

$$\mathcal{J}_n(z) = \sum_{n=0}^{\infty} (-1)^r \frac{(z/2)^{n+2r}}{\lfloor r \lfloor n+2r}$$

The function D(.) or f(.) may be used in magnetic field of blood-corpuscles, where blood is considered as two-phase pulsating and non-homogeneous magnetic field.

3.4. Radial vibration. For radial vibration of a composite spherical shell, a recent study has been given by Mahapatra [I] and before that another study on the same topic was given by Mukherjee, S. [2]. Their solutions of the problems they posed depended on the boundary conditions and basic equations they had used. In any kind of vibration, potential functions must be used. But in their studies we see that the potential functions are absent. Hence we need some work on radial vibrations in which potential functions and some new type of boundary conditions are used to solve the proposed equation of wave motion.

4. References

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