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A CHARACTERISATION OF TRANSLATION INVARIANT PROPERTIES OF FUZZY GROUPS

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Abstract: The effect of the translation operators on the chains of level subgroups of fuzzy groups was studied by the author in [6]. It was observed that some properties of fuzzy groups remain invariant under translation, while some others do not. In this paper, the author continues his investigation and gives a complete characterisation of properties which remain invariant under translation.

1. Introduction

It is a generally agreed fact that fuzziness is an unavoidable feature o most humanistic systems and it cannot be properly studied within th framework of classical set theory and two valued logic. Zadeh's[4] work o fuzzy sets provided the apt tool for dealing with fuzziness and other huma centered systems, and hence introduced a new era in mathematics. Thoug there are some controversies regarding the utility and essentiality of fuzzy s theory, it has already developed into a theory which challenges the traditional reliance on two-valued logic and classical set theory as a basis for scientific enquiry [2]. This theory has begun to be applied in multitudes of scientific areas ranging from engineering and computer science to medical diagnosis and social behaviour . The theory of fuzzy sets is , in fact, a step towards a rapproachment between the precision of classical mathematics and the pervasive imprecision of the real world [3].

There were several attempts to fuzzify various mathematical structures. The fuzzification of algebraic structures was initiated by Rosenfeld[5]. He introduced the notions of fuzzy subgroupoids and fuzzy subgroups; and obtained their basic properties. Though some other treatments of fuzzy groups are available in the literature, Rosenfeld's definition seems to be the most natural and popular one. Most of the recent works on fuzzy groups follow Rosenfeld's definition. The names of P. Bhattacharya and N. P. Mukherjee deserve special mention. In a series of papers, they have developed fuzzy parallels of several concepts in classical group theory and proved fuzzy generalisations of some important theorems like Lagrange's and Cayley's theorems.

The author's work on fuzzy groups are contained in [6] to [12]. The effect of the translation operators on fuzzy groups was studied in [6]. It was observed that some properties of fuzzy groups remain invariant under translation, while some others do not. As a continuation of this observation, here the author gives a complete characterisation of those properties of a fuzzy group which remain invariant under translation.

2. Preliminaries

A ***fuzzy subset*** a of a non-empty set X is a function $A:X \to I$, where I denotes the closed unit interval $[0,1]$ on the real line. We use the notations $\vee$ and $\wedge$ for supremum [maximum] and infimum [minimum] respectively.

Throughout this paper, G denotes an arbitrary multiplicative group with e as identity element. A fuzzy subset F of G is said to be a ***fuzzy subgroup*** of G if for every $x, y \in G$

$$F(xy) \geq \wedge \{ F(x), F(y) \} \text{ and } F(x^{-1}) = F(x).$$

It is easy to show that if F is a fuzzy subgroup of G , then

$$F(e) \geq F(x), \forall x \in G, \text{ and}$$
$$G_F = \{ x \in G : F(x) = F(e) \} \text{ is a subgroup of } G.$$

A fuzzy subgroup F of a group G is said to be ***fuzzy normal*** if

$$F(xy) = F(yx), \forall x, y \in G$$

and ***fuzzy abelian*** if G_F is an abelian subgroup of G . By ***level cardiality*** of F we mean $| I_m(F) |$.

Das[1] used Zadeh's notion of level subsets to define level subgroups of fuzzy group. Many properties of fuzzy groups have been characterised by using their level subgroups; and hence it has become one of the important tools used in the study of fuzzy groups.

For any fuzzy subset A of X and $t \in I$,

$$A_t = \{ x \in X : A(x) \geq t \}$$

is called the ***level subset*** of A at t . We shall use the notation A_t for $t \in R - I$ also in the following sense.

$$A_t = \begin{cases} \varphi, & \text{if } t > 1 \\ G, & \text{if } t < 0 \end{cases}$$

2.1. Proposition [1]. *If F is a fuzzy subgroup of G, then*

(a) $F_t = \varphi, \forall t > F(e)$ *and*

(b) F_t *is a subgroup of* $G, \forall \ 0 \leq t \leq F(e)$ □

2.2. Proposition [1]. *A fuzzy subset F of G is a fuzzy subgroup of G iff* F_t *is a subgroup of* $G, \forall \ 0 \leq t \leq F(e)$ □

If F is a fuzzy subgroup of G and $0 \leq t \leq F(e)$, then F_t is called the ***level subgroup*** of F at t.

2.3. Proposition [1]. *If F is a fuzzy subgroup of a finite grop G with $I_m(F) = \{ t_i : i = 1,2, \ldots, n\}$ then $\{ F_{t_i} : i = 1,2,\ldots,n\}$ contains all level subgroup of F. Further, if $t_1 > t_2 > \ldots\ldots t_n$ then the level subgroups form a chain.*

$$C(F) \equiv G_F = F_{t1} \subset F_{t2} \subset \ldots\ldots\ldots \subset F_{tn} = G \quad \square$$

The above proposition considers only fuzzy subgroups of finite groups. This restriction is removed in the following proposition.

2.4. Proposition [6]. *Let F be a fuzzy subgroup of an arbitrary group G with $I_m(F) = \{ t_j : j \in J\}$ and $\mathfrak{I} = \{ F_{t_j} : j \in J\}$, J being an rbitrary index set. Then*

(a) $\exists$ *a unique* $j_0 \in J$ *such that* $t_{j_0} \geq t_j$, $\forall j \in J$

(b) $G_F = \bigcap_{j \in J} F_{t_j} = F_{t_{j_0}}$

(c) $G = \bigcup_{j \in J} F_{t_j}$ *and*

(d) *the members of $\mathfrak{I}$ form a chain* $\square$

It may be observed that, in the finite case, $\mathfrak{I}$ is the chain, of ***all*** level subgroups of F. But, in the general case, $\mathfrak{I}$ need not contain all level subgroups of F.

The ***translation operators*** $T_{\alpha+}$ and $T_{\alpha-}$ for a fuzzy subset A of a nonempty set X are defined as follows:

$$T_{\alpha+}(A)(x) = \wedge \{ A(x) + \alpha, 1 \}$$
$$T_{\alpha-}(A)(x) = \wedge \{ A(x) - \alpha, 0 \}, \forall x \in X, \alpha \in I.$$

It can be easily verified that $T_{\alpha+}$ and $T_{\alpha-}$ are, in general, not inverse operators. That is,

$$T_{\alpha-}[T_{\alpha+}(A)] \neq A \text{ and } T_{\alpha+}[T_{\alpha-}(A)] \neq A$$

But, $\quad T_{\alpha-}[T_{\alpha+}(A)] = A \Leftrightarrow \alpha\ 1 - A_\vee$ and

$$T_{\alpha+}[T_{\alpha-}(A)] = A \Leftrightarrow \alpha \leq A_\wedge$$

where $\quad A_\vee = \vee \{ A(x) : x \in X \}$ and
$A_\wedge = \wedge \{ A(x) : x \in X \}$

We use the notations

$$F^{\alpha+} = T_{\alpha+}(F), \quad F^{\alpha-} = T_{\alpha-}(F)$$

and $F_t^{\alpha+}$, $F_t^{\alpha-}$ for their level subgroups.

2.5. Proposition[6]. *The chain of level subgroups of F, $F^{\alpha+}$ and $F^{\alpha-}$ are related as follows.*

$$C(F^{\alpha+}) \subseteq C(F) \text{ and } C(F^{\alpha-}) \subseteq C(F)$$

But if $\alpha < \wedge \{ 1 - \vee \{ t_j : j \neq j\}, \wedge \{ t_j : j \neq j_0\}\}$ then

$$C(F^{\alpha+}) \equiv C(F) \equiv C(F^{\alpha-}) \quad \square$$

2.6. Proposition [6]. *The following are equivalent:*

a) *F is a fuzzy subgroup of G*
b) *$F^{\alpha+}$ is a fuzzy subgroup of G, $\forall \alpha \in I$*
c) *$F^{\alpha-}$ is a fuzzy subgroup of G, $\forall \alpha \in I$* $\square$

It follows from the above proposition that ***being a fuzzy subgroup*** is a translation invariant property. It can be proved that fuzzy normality also is translation invariant; but fuzzy abelianness is , in general, not translation invariant.

3. Characterisation of translation invariant properties

3.1. Definition. Let F be a fuzzy subgroup of a group G and P be a property of F. We say that P is ***characterisable on level subgroups*** if the characteristic function χ_{F_t} of F_t has property P, $\forall\ t \in [0, F(e)]$, whenever F has the property P.

3.2. Examples. It follows from proposition 2.2 that 'being a fuzzy subgroup' is a property which is characterisable on level subgroups. Fuzzy normality is another obvious example. $\square$

3.3. Definition. A property of a fuzzy group is said to be ***translation invariant*** if F has property P $\Rightarrow$ all translates of F also have the property.

3.4. Proposition. *If P is a translation variant property of a fuzzy subgroup, then χ_{G_F} has property P whenever F has it .*

Proof. Straight forward. □

3.5. Theorem. *A property of a fuzzy subgroup is translation invariant iff it is characterisable on level subgroups.*

Proof. *Necessity*: Suppose P is a translation invariant property of fuzzy groups. We have to show that P is characterisable on level subgroups.

Let F be a fuzzysubgroup of a group G having property P. Choose $t \in [0, F(e)]$ arbitrarily and then fix it. Consider the level subgroup F_t. It follows from proposition 2.1.(b) that χ_{F_t} fuzzy subgroup of G. We want to prove that χ_{F_t} has property P.

Let $\beta = F(e) - t$

By proposition 2.6, $F^{\beta-}$ is a fuzzy subgroup of G .Since F has property P, by the hypothesis, $F^{\beta-}$ also has property P. Hence, by proposition 3.4. the characteristic function of $G_{F^{\beta-}}$ is a fuzzy subgroup of G having property P. Now,

$$G_{F^{\beta-}} = \{ x \in G : F^{\beta-}(x) = F^{\beta-}(e) \} = F_t^{\beta-}$$

Hence it follows that $\chi_{F_t} = \chi_{G_{F^{\beta-}}}$ has the property P. This proves the necessary part.

Sufficiency: Suppose P is characterisable on level subgroups. We want to prove that P is translation invariant.

Let F be any fuzzy subgroup of a group G having property P. By the hypothesis, χ_{F_t} is a fuzzy subgroup of G having property P; $\forall t \in [0, F(e)]$.

Let $\alpha \in [0, 1]$ and $F_1 = T_{\alpha+}(F)$ and $F_2 = T_{\alpha-}(F)$ .It follows from proposition 2.5 that the level subgroups of F_1 and F_2 are all or some of those of F.

$\therefore$ F has property P

$\Rightarrow \chi_{F_t}$ has property P, $\forall\, t \in [0, F(e)]$

$\Rightarrow F_t$ has the property *crisp P*, $\forall t \in [\,0, F(e)\,]$

$\Rightarrow$ All level subgroups of F_1 and F_2 has property *crisp P* ,$\forall t \in [0,F(e)]$

$\Rightarrow F_1$ and F_2 have property P

$\therefore$ P is translation invariant . Hence the theorem. □

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